Chapter 12. Introduction to Dynamic Optimization:
The Calculus of Variations

Static models aim to find values of the independent variables that maximize particular functions. Such optimization problems seek the value or values of an argument that optimize a given function at a particular point. Dynamic models aim to find not just the maximum value of some function, but rather, the actual function that provides a time path for the values of the economic variables so that some value function is maximized or minimized over a given interval of time. In dynamic optimization, we try to find a curve \( y'(t) \) that will maximize or minimize a given integral. The integral, as we know, gives the area under a curve \( F \) which is a function of the independent variable \( t \), the function \( y(t) \), and its derivative \( \frac{dy}{dt} \) or \( y'(t) \). Note that the independent variable \( t \) denotes time, which is why we speak of dynamic optimization. Therefore, if we assume a time period from \( t_o \) (usually zero) to \( t_1 \), the dynamic optimization problem is to maximize or minimize the integral expression

\[
I = \int_{t_o}^{t_1} F[t, y(t), y'(t)] dt
\]

where the function \( F(t, y, y') \) is assumed to be continuous for \( t, y(t), \) and \( y'(t) \) and to be differentiable, that is, to have continuous partial derivatives with respect to \( y \) and \( y' \). Here \( t_o, t_1, y_o \) and \( y_1 \) are given parameters. An integral that assumes a numerical value for each of the class of functions \( y(t) \) is called a functional. As opposed to ordinary calculus that deals with functions, the calculus of variations is a special field of mathematics that deals with functionals. Those are generally integrals involving an unknown function and its derivatives. We refer to the integral \( I \) as a functional because it is a function of the functions \( y(t) \) and \( y'(t) \), but we are more interested in an extremal, the function that finds the maximum or minimum value of the functional. More specifically, an extremal is the curve that optimizes the value of the functional. In order for the class of functions \( y(t) \) to be extremals, they should be continuously differentiable on the defined interval and should satisfy some fixed endpoint (boundary) conditions.

Perhaps the simplest example of such an optimization problem is to find the length of a nonlinear curve giving the shortest distance between two points on a plane. Such two points are \( (t_o, y_o) \) and \( (t_1, y_1) \) where we have the function \( y = f(t) \). Although nonlinear, the distance between them can be approximated easily using the Pythagoras theorem. Given the diagram in Figure 1, for very small distances \( dt, dy \) and \( ds \) we have the dependence

\[
(ds)^2 = (dt)^2 + (dy)^2
\]

\[
ds = \sqrt{(dt)^2 + (dy)^2}
\]

Factoring out the term \( dt \) from the right side,

\[
ds = \sqrt{1 + \left(\frac{dy}{dt}\right)^2} \ dt
\]
\[ ds = \sqrt{1 + (y')^2} \, dt \]

Summing up all the distances, we obtain the arc length of the entire curve from point \( t_o \) to \( t_1 \) as

\[ A[y] = \int_{t_o}^{t_1} \sqrt{1 + (y')^2} \, dt \]

Furthermore, to find the shortest distance between these two points, we have to minimize the integral found.

**Figure 1**

Dynamic optimization studies the optimal time path of a particular function and often deals with stock-flow relationships among the variables at successive points in time. Some of the variables involved are stock concepts, also called state variables in dynamic optimization, while flow concepts are often referred to as control variables. For instance, in the context of production theory, stocks change from one period to another and their increase depends on both the stocks and flows within this interval.

With optimization over time the objective function can be expressed as the sum, difference or product of functions that are also changing over time. For example, a firm maximizing the present value of its stream of revenues would account for its total revenue but would also consider the interest rate \( r \) as the discount factor. With optimal time path the optimization problem usually begins with an initial moment \( t_o \) and ends at a finite moment \( t_1 \). The initial state variable \( y_o \) is taken as given and, in addition, some terminal condition is specified. More specifically, for the firm trying to maximize its stream of revenues \( R \) from time \( t_o \) to \( t_1 \) it may be that this stream depends on the own price of the product \( p \) and on the rate of change of price with respect to time \( p'(t) \). Thus, the optimization problem for the firm can be written as

\[
\max_{t_o} \int_{t_o}^{t_1} R(t, p(t), p'(t)) e^{-rt} \, dt
\]

subject to \( p(t_o) = p_o \) and \( p(t_1) = p_1 \)

where total revenue is discounted at the interest rate \( r \) and the two constraints, the initial and the terminal one, are the boundary conditions.

**Euler’s Equation**

The mathematical problem of finding a function that minimizes or maximizes some integral got its systematic solution by Leonhard Euler and Joseph Louis Lagrange\(^1\) who in the 1750s first introduced a general differential equation necessary to solve such problems. This lay the foundation of the

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\(^1\) Leonhard Paul Euler (1707-1783) and Joseph-Louis Lagrange (1736-1813).
calculus of variations, which seeks to find a curve, path, or surface that gives an optimum (or stationary) value for a given function.

In the 1950s, L. S. Pontryagin and his colleagues in the Soviet Union developed optimal control theory, a special branch of which is the classical calculus of variations. In parallel with Pontryagin, whose focus was on the physical sciences, a team of scholars led by Richard Bellman developed dynamic programming for the purpose primarily of economics and management science. In view of the advanced level and rigor of optimal control theory and dynamic programming, which go beyond the scope and aims of this book, we will cover only the simple techniques of the calculus of variations and take a brief glance at optimal control theory. Although the three approaches have different relevance to, and usefulness in, analytical economics, they all lead to the same solution.2

The so-called Euler’s equation gives a necessary condition for dynamic optimization. It is a differential equation for the solutions of which a given functional is stationary. In order for the curve connecting two points \((t_o, y_o)\) and \((t_1, y_1)\) to qualify as an extremal, that is, to optimize the functional

\[
I = \int_{t_o}^{t_1} F[t, y(t), y'(t)] dt
\]

\[y(t_o) = y_o, \quad y(t_1) = y_1\]

a necessary but not a sufficient condition is that

\[
\frac{\partial F}{\partial y} = \frac{d}{dt} \left( \frac{\partial F}{\partial y'} \right)
\]

which represents the Euler’s equation. Alternatively, the equation can be written in the form

\[
F_y(t, y, y') = \frac{d}{dt} \left[ F_{yy}(t, y, y') \right]
\]

or simply

\[
F_y = \frac{dF_{yy}}{dt}
\]

and, given that \(t, y\) and \(y'\) are all functions of \(t\), by taking the total derivative of the right-hand side with respect to \(t\) and using the chain rule, we obtain

\[
F_y = F_{yt} + F_{yt'} y' + F_{yty} y''
\]

where \(y'' = \frac{d^2 y}{dt^2}\). The differential equation we obtain is of the second order. The exact way to solve the Euler’s equation is illustrated best with numerical examples, which follow later in the chapter. Not every curve \(I(t, y, y')\) connecting two points is suitable for an extremal. In order to find such a curve that optimizes a given functional subject to some fixed boundary conditions in dynamic optimization, we just follow several simple steps:

1. For the integrand \(F = F(t, y, y')\), we take the partial derivatives of \(F\) with respect to \(y\) and \(y'\) or \(F_y\) and \(F_{y'}\).
2. We substitute these two values in the Euler’s equation \(F_y = \frac{dF_{y'}}{dt}\). 
3. Then we take the derivative of \(F_{y'}\) with respect to \(t\).
4. In the absence of any derivatives such as \(y'\) or \(y''\), we solve directly for \(y\). If there are such terms, we integrate until all the derivatives disappear and again we solve for \(y\).

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This does not yet prove that we indeed have the needed maximum or minimum. In the elementary calculus of a single variable we resort to the second derivative \( y''(t) \) as a test for the extremum and check whether it is positive or negative which implies a minimum or a maximum for \( y(t) \), respectively. The calculus of variations is not interested in small changes of the argument \( t \) but, rather, in the integral of the function \( F(t, y, y') \). Just as with second-order conditions in static optimization, we have second-order conditions for dynamic optimization problems that allow us to determine a maximum or a minimum, but deeper knowledge of the subject is needed. In the case of shortest distance given by the arc length \( A[y] \), the integral \( I(t, y, y') \) is a functional and we formulate the task of the minimization problem as finding a minimum in the \( I \)-space, given the parameters \( y \) and \( y' \). Since we can always find a curve the path’s length of which is greater than some finite path length \( I_o \), the stationary point found would represent a minimum. We now prove that since the slope of the curve giving the shortest path between two points is a constant, this curve must be a straight line. For the arc length of the curve connecting the two points \((t_o, y_o)\) and \((t_1, y_1)\), we found

\[
A[y] = \int_{t_o}^{t_1} \sqrt{1 + (y')^2} \, dt
\]

where the integrand is \( F(t, y, y') = \sqrt{1 + (y')^2} \) evaluated at \([t, y(t), y'(t)]\). The partial derivatives of \( F \) are

\[
F_y = \frac{\partial F}{\partial y} = 0 \quad \text{and} \quad F_{y'} = \frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{1 + (y')^2}}
\]

Substituting in the Euler’s equation,

\[
\frac{\partial F}{\partial y} = \frac{d}{dt} \left( \frac{\partial F}{\partial y'} \right)
\]

\[
\frac{d}{dt} \left( \frac{y'}{\sqrt{1 + (y')^2}} \right) = 0
\]

or \( F_{y'} = \frac{y'}{\sqrt{1 + (y')^2}} = c \) must be a constant in order for its derivative with respect to \( t \) to be zero. But if the parental function \( F_y \) has a constant slope of \( c \) given by its first derivative \( F_{y'} \), then it must be a straight line. Hence, a straight line would always give the shortest distance between two points.

**Example:** Given the functional \( \int_0^1 \left[ 24yt + 2(y')^2 - 4t \right] dt \) subject to \( y(0) = 1 \) and \( y(1) = 3 \), find \( y(t) \) following the conditions for dynamic optimization.

1. Since the integrand is \( F = 24yt + 2(y')^2 - 4t \), we have

\[
F_y = 24t \quad \text{and} \quad F_{y'} = 4y'
\]

2. Substituting in Euler’s equation,

\[
24t = \frac{d}{dt} (4y')
\]

3. Since \( \frac{d}{dt} \left( \frac{dy}{dt} \right) = \frac{d^2 y}{dt^2} = y'' \), we have
24t = 4y''
4. We have the second-order derivative $y''$, and in order to drop it we integrate both sides:

$$\int 24dt = \int 4y''dt$$

$12t^2 + c_1 = 4y'$

And integrating further,

$$\int (12t^2 + c_1)dt = \int 4y'dt$$

$4t^3 + c_1t + c_2 = 4y$ which gives

$$y = t^3 + \frac{c_1}{4} + \frac{c_2}{4}$$

With the help of the boundary conditions, we can definitize the constants $c_1$ and $c_2$.

$y(0) = \frac{c_2}{4} = 1$ or $c_2 = 4$

$y(1) = 1 + \frac{c_1}{4} + 1 = 3$ or $c_1 = 4$

So, finally for $y$,

$$y(t) = t^3 + t + 1$$

The Ramsey Growth Model

Frank Ramsey in 1928 developed a neoclassical growth model alternative to that advanced by Solow. Neoclassical growth models generally assume that gross investment is the difference between national income and aggregate consumption. Given that gross investment is also the sum of net investment and depreciated capital, we have

$$I_g = I_n + \rho K = K' + \rho K$$

and

$$I_g = Y(K) - C$$

or net investment can be expressed as

$$K' = Y(K) - \rho K - C$$

where consumption, investment and capital stock are all assumed to be functions of time. If the variables are taken on a per worker basis, the equation transforms into

$$k' = f(k) - \rho k - c$$

which represents the fundamental differential equation of neoclassical economic growth. Here $k$ is the well known capital-labor ratio, $f(k)$ is output per worker, $c$ is consumption per worker and $\rho$ is the depreciation rate that can take values between 0 and 1. Net investment or the increase in capital per worker thus is the part of output saved less the depreciation in capital. Seeking an optimal time path for economic growth neoclassical growth models express utility as a function of consumption, either

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4 Accounting for the growth rate of the labor force $\lambda$ the equation becomes $k' = f(k) - (\lambda + \rho)k - c$. 
aggregate $U[C(t)]$ or individual $U[c(t)]$, and maximize dynamically the discounted value of this utility function. For instance, the optimization problem could be

$$\max \int_{t_0}^{t_1} U[C(t)] e^{-rt} dt$$

where the discount rate is $r$ and utility is maximized from moment $t_0$ to $t_1$. This allows obtaining an optimal time path for capital stock $K(t)$ and, from there on, of consumption, labor force, investment, etc. Being familiar with the Solow growth model, we already know that in order for the economy to be in a steady state all those variables should grow at the same rate. Recall the outcome of the Solow model that if population grows at the rate $\lambda$ as in the exponential function $L(t) = L_0 e^{\lambda t}$, investment, capital, etc. should also grow at that rate in order for the economy to be stable and not on the “razor’s edge” (as the Domar growth model predicted).

Like Solow, Ramsey studies the capital-labor ratio $k$ and its time path. However, in addition to $k$, Ramsey analyzes the saving ratio $s$, giving the share of investment in aggregate output $Y$ or

$$s(t) = \frac{I}{Y} \quad 0 < s < 1$$

Unlike Solow, Ramsey assumes that the saving ratio changes with time and the convergence of the economy to its steady state is not uniform. This assumption transforms the saving rate from exogenous into endogenous and the task is to find its optimal time path. The model concludes that at the optimum the saving rate is at its golden rule level, that is, the level that maximizes the sustainable level of consumption per worker. Thus, the model is consumption oriented, rather than production oriented. A convenient class of utility functions consistent with the steady state of the economy is the class of functions known as CRRA functions of the type $U[C(t)] = \frac{[C(t)]^{1-\eta}}{1-\eta}$ such that the marginal utility of consumption is $U' = C^{-\eta}$. Thus, the optimization problem becomes

$$\max \int_{t_0}^{t_1} U[C(t)] e^{-rt} dt = \int_{t_0}^{t_1} \frac{[C(t)]^{1-\eta}}{1-\eta} e^{-rt} dt$$

Households maximize their consumption dynamically, whereby at each point in time they set their intertemporal marginal benefit of consumption equal to their marginal cost. Furthermore, they set optimal consumption at the level where it pays to substitute present consumption with future. Both decisions depend on the optimal result that obtains

$$\frac{C'}{C} = \frac{\rho - r}{\eta}$$

by which the growth rate of consumption is the product of the intertemporal elasticity of substitution $\frac{1}{\eta}$ between consumption in two periods and the difference of the interest rate $\rho$ and the intertemporal discount rate $r$. While the interest rate $\rho$ is the reward for postponing consumption and stimulates its growth, $r$ is the cost of consuming presently and reduces consumption. A high elasticity of

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5 We should also note that initially Ramsey designed the model as a central-planner’s problem of optimizing the consumption levels intertemporally, that is, over successive generations. Only later was the model developed to fit a decentralized dynamic economy.

6 CRRA stands for constant relative risk aversion functions for which $U[C(t)] = \frac{[C(t)]^{1-\eta}}{1-\eta}$ if $\eta \neq 1$ and $U[C(t)] = \ln C(t)$ for $\eta = 1$. 
substitution $\frac{1}{\eta}$ also stimulates the growth of consumption. Both the elasticity of substitution and the trade-off between the interest and the discount rate are mechanisms working in the direction of the substitution effect, that is, convincing the consumer that future consumption is a good substitute for current consumption.

### A Cost-Minimizing Firm

A firm’s total costs depend on the level of output $q(t)$ and its rate of change with time $q'(t)$ according to the function

$$TC = aq + b(q')^2$$

$a, b > 0$

The firm wishes to minimize its costs of production where the endpoint conditions are $q(0) = 0$ and $q(2) = \frac{3a}{2b}$. Formulate the dynamic optimization problem and find a candidate for an extremal that will minimize the firm’s costs. The optimization problem is

$$\min \int_0^2 \left[aq + b(q')^2\right]dt$$

subject to $q(0) = 0$ and $q(2) = \frac{3a}{2b}$

The functional, therefore, is $F[t, q(t), q'(t)] = [aq + b(q')^2]$. We can easily find the partial derivatives $F_q = a$ and $F_{q'} = 2bq'$. Substituting in the Euler’s equation,

$$a = \frac{d}{dt} (2bq')$$

which transforms into

$$a = 2bq''$$

or

$$q'' = \frac{a}{2b}$$

We can easily solve for $q(t)$ by integrating twice. This will give

$$q' = \frac{at}{2b} + c_1$$

and integrating once more,

$$q(t) = \frac{at^2}{4b} + c_1t + c_2$$

To definitize the constants, we use the endpoint conditions

$$q(0) = \frac{a(0)}{4b} + c_1(0) + c_2 = 0$$

or $c_2 = 0$

$$q(2) = \frac{a(2)^2}{4b} + c_1(2) + 0 = \frac{3a}{2b}$$

or $c_1 = \frac{a}{4b}$

Thus, the candidate for extremal is

$$q(t) = \frac{at^2}{4b} + \frac{at}{4b}$$
We now need to prove that the output level found is indeed a minimum, which requires further knowledge. As we can easily see, the output function of the firm is increasing with time.

**Constrained Dynamic Optimization**

So far, we have optimized unconstrained functionals. Often, though, we may have to apply the techniques of calculus of variations and optimize an integral subject to some functional constraint. Given the integral

\[ I = \int_{t_0}^{t_f} F[t, y(t), y'(t)] dt \]

\[ y(t_0) = y_0 \quad y(t_f) = y_1 \]

we may seek an extremal that maximizes or minimizes it subject to the constraint

\[ \int_{t_0}^{t_f} C[t, y(t), y'(t)] dt = c \]

where \( c \) is a constant. Problems such as this, where the constraint is an integral that is held constant, are known as isoperimetric problems. Here \( C(t, y, y') \) is a continuously differentiable function in the given parameters. There could as well be constrained dynamic optimization problems where the constraints are one or several equality or inequality constraints connecting the state variables, their rate of change and time. We shall deal only with integral constraints in the examples that follow. Similar to static optimization, we can apply the Lagrange multiplier method. In order to do that, we multiply the constraint by \( \lambda \) and add it to the objective function such that

\[ \int_{t_0}^{t_f} (F + \lambda C) dt \]

A necessary, though not a sufficient, condition to have an extremal for dynamic optimization is the Euler-Lagrange equation where

\[ \frac{\partial L}{\partial y} = \frac{d}{dt} \left( \frac{\partial L}{\partial y'} \right) \quad \text{and} \quad L = F + \lambda C \]

Constrained optimization of functionals is often used in the calculus of variations to find a curve the perimeter of which encloses the largest area.

**Example:** Find the curve of a given length \( m \) that encloses a maximum area \( A \) given by the expression

\[ A = \frac{1}{4} \int (ty' - y) dt \]

where the length of the curve is

\[ \int_{t_0}^{t_f} \sqrt{1 + (y')^2} dt = m \]

Following the steps of dynamic optimization subject to functional constraints, we form the Lagrangian.

\[ \int_{t_0}^{t_f} \left[ \frac{1}{4} (ty' - y) + \lambda \sqrt{1 + (y')^2} \right] dt = m \]
Therefore, the integrand is  
\[ L = \frac{1}{4} (ty' - y) + \lambda \sqrt{1 + (y')^2} \]  
and
\[ \frac{\partial L}{\partial y} = -\frac{1}{4} \quad \quad \frac{\partial L}{\partial y'} = \frac{1}{4} t + \frac{\lambda y'}{\sqrt{1 + (y')^2}} \]

Substituting in the Euler-Lagrange equation,

\[ \frac{\partial}{\partial y'} \left( \frac{d}{dt} \left( \frac{\partial L}{\partial y'} \right) \right) - \frac{1}{4} = d \left( \frac{1}{4} t + \frac{\lambda y'}{\sqrt{1 + (y')^2}} \right) \]

\[ -\frac{1}{4} = \frac{d}{dt} \left[ \frac{1}{4} + \frac{\lambda y'}{\sqrt{1 + (y')^2}} \right] \]

\[ \frac{d}{dt} \left[ \frac{\lambda y'}{\sqrt{1 + (y')^2}} \right] = -\frac{1}{2} \]

We integrate both sides with respect to \( t \), which gives

\[ \frac{\lambda y'}{\sqrt{1 + (y')^2}} = -\frac{1}{2} (t - c_i) \]

Raising both sides of the equation to the second power and rearranging,

\[ 4\lambda^2 (y')^2 = (t - c_i)^2 \left[ 1 + (y')^2 \right] \]

\[ 4\lambda^2 (y')^2 - (t - c_i)^2 (y')^2 = (t - c_i)^2 \]

\[ (y')^2 = \frac{(t - c_i)^2}{4\lambda^2 - (t - c_i)^2} \]

\[ y' = \pm \frac{t - c_i}{\sqrt{4\lambda^2 - (t - c_i)^2}} \]

Integrating again both sides where on the right we use integration by substitution and set \( u = (t - c_i)^2 \) and, therefore, \( \frac{du}{dy} = 2(t - c_i) \), we obtain

\[ y - c_2 = \pm \sqrt{4\lambda^2 - (t - c_i)^2} \]

\[ (t - c_i)^2 + (y - c_2)^2 = 4\lambda^2 \]

which is an expression for a circle and the parameters \( c_i, c_2 \) and \( \lambda \) are determined with the help of the endpoint conditions \( t_o \) and \( t_i \) and the constant \( m \).

**A Glimpse of Optimal Control Theory**

Dynamic optimization finds application in economics when it comes to allocating scarce resources over competing purposes within a given time interval (say, between the initial moment \( t_o \) and the
Optimal control theory seeks to find optimal time paths for control (flow) variables among a class of optimal time paths called a control set. The optimal time path for the control variable is chosen with the help of a set of differential equations known as the equations of motion. This optimal time path further determines a time path for the state (stock) variables describing the system. The time path of the control variable is such that it maximizes a given functional formulated in accordance with the optimization problem and set by the time paths of both the control and the state variables. Optimizing such an objective functional equivalent to an extremal in the calculus of variations gives the so-called control problem.

A simple control problem would involve time, state variables, control variables, the equations of motion, the endpoint (transversality) conditions, and the extremal (objective functional). The general form of the control problem would be

$$\max \int_{t_o}^{t_1} F[t, y(t), u(t)] dt$$

subject to $y'(t) = f[t, y(t), u(t)]$ with endpoint conditions $y(t_o) = y_o$ and $y(t_1) = y_1$

This looks very similar to the problem of the classical calculus of variations. In fact, the calculus of variations problem can be considered a special case of the general control problem in which there is only one state variable and one control variable and the control variable is simply the rate of change of the state variable with time rather than a more general function involving the state variable or time as well. Thus, the equation of the motion (or state equation) for the calculus of variations problem is

$$y' = f(t, y, u) = u$$

To solve the general control problem, given the constraint, we have

$$\max F(t, y, u) + \mu f(y, u) + y \mu'$$

where $\mu$ is a Lagrange multiplier that takes care of the constraint given by the equation of the motion. Thus formulated, the control problem gives rise to the following Hamiltonian

$$H = F + \mu f$$

and optimum equations

$$H_u = \frac{\partial H}{\partial u} = F_u + \mu f_u = 0 \quad \text{(maximum principle)}$$

$$H_y = \frac{\partial H}{\partial y'} = F_y + \mu f_y = -\mu' \quad \text{or} \quad F_y + \mu f_y + \mu' = 0 \quad \text{(costate equation)}$$

The first equation sets the so-called maximum principle of optimum control theory, while the second is called a costate or adjoint equation. These equations serve as first-order conditions for optimizing the control problem. They, together with the state equation, are the necessary conditions for the optimal path of the state and control variables over the given time span.

To illustrate how a calculus of variations problem can be converted into a control problem, we use the fundamental equation of neoclassical growth models discussed previously

$$K'(t) = Y[K(t)] - \rho K(t) - C(t)$$

We know from our first economics classes that capital stock $K(t)$ is a stock variable, while its time rate of change investment $K'(t)$ is a flow variable. In the context of dynamic optimization, capital stock is a state variable, while consumption affected by investment is the control variable. Furthermore, with a utility function $U[C(t)]$ (assuming diminishing marginal utility or $U''(C) < 0$), where the rate of discount is $r$, the optimization problem of the calculus of variations would be
max \( \int_{t_o}^{t_f} U[C(t)] e^{-\rho t} dt \) 

subject to \( K'(t) = Y[K(t)] - \rho K(t) - C(t) \) where \( K(t_o) = K_o \) and \( K(t_f) = K_f \). This gives rise to the Hamiltonian

\[
H = U(C) e^{-\rho t} + \mu [Y(K) - \rho K - C]
\]

with the following maximum and adjoint equations:

\[
H_C = U'(C) e^{-\rho t} - \mu = 0
\]

\[
H_K = \mu (Y_K - \rho) = -\mu'
\]

Differentiating the maximum principle equation further with respect to \( t \) gives

\[
U' C e^{-\rho t} - r U e^{-\rho t} - \mu' = 0 \quad \text{where} \quad -\mu' = \mu (Y_K - \rho) = U' e^{-\rho t} (Y_K - \rho)
\]

\[
U' C e^{-\rho t} - r U' e^{-\rho t} + U' e^{-\rho t} (Y_K - \rho) = 0
\]

\[
U' C' - r U' + U' (Y_K - \rho) = 0
\]

Thus, at the optimum, we have

\[
\frac{U' C'}{U'} = r + \rho - Y_K
\]

The left-hand side again gives the proportional increase with time of the marginal utility of consumption. Being the marginal benefit of additional consumption at any point in time, it must be equal to the marginal cost of increasing consumption given here by the discount rate \( r \), the depreciation rate \( \rho \), and the loss of additional aggregate output given here by the marginal product of capital \( Y_K \). Finally, the assumption that the nation experiences diminishing marginal utility or \( U''(C) < 0 \) implies a concave utility function. In the language of optimal control theory, this means that the Hamiltonian is concave in the state and control variables, which ensures that the solution of this control problem is indeed a maximum.

Now that we are somewhat familiar with the tools of optimal control, we can go back to the classical calculus of variations and provide an elegant proof to the Euler’s equation. Transformed in calculus of variations terms, the control problem becomes

\[
\max \int_{t_o}^{t_f} F[t, y(t), y'(t)] dt
\]

where \( y'(t) = f[t, y(t), u(t)] = u \) and \( y(t_o) = y_o \) and \( y(t_f) = y_f \)

Thus, the Hamiltonian is

\[
H = F + \mu f = 0
\]

\[
H_u = H_y = F_y' + \mu f_y' = 0 \quad \text{(maximum principle)}
\]

\[
H_y = F_y + \mu f_y' + \mu' = 0 \quad \text{(costate equation)}
\]

Since \( y'(t) = f[t, y(t), u(t)] = u \), we have \( f_{y'} = 1 \) and \( f_y = 0 \). Thus, the two equations reduce to
Chapter 12. Introduction to Dynamic Optimization: The Calculus of Variations

\[ F_{y'} = -\mu \quad \text{and} \quad F_y = -\mu' \]

Since \( -\mu' = \frac{d(-\mu)}{dt} \), we must have \( F_y \) as the time derivative of \( F_{y'} \), that is, \( F_y = \frac{dF_{y'}}{dt} \), which is nothing but the Euler’s equation or \( \frac{\partial F}{\partial y} = \frac{d}{dt} \left( \frac{\partial F}{\partial y'} \right) \). Hence, we obtain that an important condition for finding an optimal time path for the state and control variables and solving a dynamic optimization problem using the calculus of variations is for the Euler’s equation to hold.

Problems

1. A firm’s total costs of production depend on the level of output \( q(t) \) and its rate of change with time \( q'(t) \) according to the function \( TC = aq^2 - bq' \) \((a, b > 0)\). The firm wishes to minimize the present value of its costs of production where the discount rate is \( r \) and the costs would be incurred at time \( t_1 \). Find a candidate for the extremal that will minimize the firm’s costs.

Solution:

We have

\[ \min_{t_0} \int_{t_0}^{t_1} (aq^2 - bq') e^{-rt} dt \]

The functional is \( F[t, q(t), q'(t)] = (aq^2 - bq') e^{-rt} \). We have the partial derivatives

\[ F_q = 2aqe^{-rt} \quad \text{and} \quad F_{q'} = -be^{-rt} \]

Substituting in the Euler’s equation,

\[ 2aqe^{-rt} = \frac{d}{dt} (-be^{-rt}) \]

\[ 2aqe^{-rt} = rb e^{-rt} \]

\[ q(t) = \frac{rb}{2a} \]

We see that the suitable candidate for an extremal is a constant function of time that depends positively on the discount rate.

2. A monopolist is faced with the following demand function \( q(t) = ap(t) + bp'(t) + c \) where the number of units produced by the monopolist \( q(t) \) depends both on the price of the good \( p(t) \) and its rate of change with time \( p'(t) \). The firm wishes to maximize the present value of its future stream of revenues from the sale of its product. Assuming the discount rate is \( r \), find the suitable pricing policy for this monopolist.

Solution:

We have the problem

\[ \max_{t_0} \int_{t_0}^{t_1} p(t)q(t)e^{-rt} dt \]

Solving further,

\[ \int_{t_0}^{t_1} p(t)q(t)e^{-rt} dt = \int_{t_0}^{t_1} p(t)[ap(t) + bp'(t) + c] e^{-rt} dt = \int_{t_0}^{t_1} (ap^2 + bpp' + cp)e^{-rt} dt \]

The functional, therefore, is \( F[t, p(t), p'(t)] = (ap^2 + bpp' + cp)e^{-rt} \) and
Substituting in the Euler’s equation,

\[ (2ap + bp' + c)e^{-rt} = 2ape^{-rt} + ce^{-rt} - rbpe^{-rt} \]

or

\[ 2ap + c = -rbp \]

and

\[ p(t) = \frac{c}{2a + rb} \]

gives the proper pricing policy for the monopolist.

3. A firm wishes to minimize the discounted value of its total costs of producing \( n \) units to be delivered at time \( t_i \) where the discount rate is \( r \). The costs of the firm are given by the function \( TC = aq + b(q')^2 \) where \( q(t) \) is the level of output and \( q'(t) \) is its rate of change with time and \( a \) and \( b \) are constants. Find an output function of time that is a suitable candidate for an extremal minimizing the firm’s costs.

Solution:

The optimization problem is

\[
\min \int_{t_{0}}^{t_{i}} \left[ aq + b(q')^2 \right] e^{-rt} \, dt
\]

subject to \( q(t_{0}) = 0 \) and \( q(t_{i}) = n \)

The functional is \( F[t, q(t), q'(t)] = \left[ aq + b(q')^2 \right] e^{-rt} \) and the partial derivatives are

\[
F_q = ae^{-rt} \quad \text{and} \quad F_{q'} = 2bq'e^{-rt}
\]

Substituting in the Euler’s equation,

\[
ae^{-rt} = \frac{d}{dt} (2bq'e^{-rt})
\]

or

\[
ae^{-rt} = 2bq''e^{-rt} - 2rbq'e^{-rt}
\]

\[
a = 2bq'' - 2rbq'
\]

Rearranging and normalizing the equation gives

\[
q''(t) - rq'(t) = \frac{a}{2b}
\]

which is a second-order differential equation. We can solve it using the formula for the particular integral and the complementary function of a second-order linear equation. For the parameters, we have \( a_1 = -r \), \( a_2 = 0 \) and \( c = \frac{a}{2b} \). Thus, for the particular integral, we have

\[
q_{p} = \frac{c}{a_1} t = -\frac{a}{2br} t
\]

We also need to find the complementary function which gives two characteristic roots using the well-known formula.
Thus, \( r_1 = 0 \) and \( r_2 = 1 \). According to the formula,

\[
q(t) = A_1 e^{rt} + A_2 e^{st} + q_p
\]

or

\[
q(t) = A_1 e^{rt} + A_2 - \frac{a}{2} t
\]

To definitize the constants, we use the constraints \( q(t_0) = 0 \) and \( q(t_1) = n \). Substituting \( t_0 = 0 \) and \( t_1 \)

\[
q(0) = A_1 + A_2 = 0 \quad \text{and} \quad q(t_1) = A_1 e^{r t_1} + A_2 - \frac{a}{2} t_1 = n
\]

\[
A_1 e^{r t_1} - \frac{a}{2} t_1 = n
\]

\[
A_1 (e^{r t_1} - 1) = n + \frac{a}{2} t_1
\]

\[
A_1 = \frac{n + \frac{a}{2} t_1}{e^{r t_1} - 1} \quad \text{and} \quad A_2 = -\frac{n + \frac{a}{2} t_1}{e^{r t_1} - 1}
\]

which gives the final form of the output function

\[
q(t) = \left( \frac{n + \frac{a}{2} t_1}{e^{r t_1} - 1} \right) e^{r t} - \left( \frac{n + \frac{a}{2} t_1}{e^{r t_1} - 1} \right) - \frac{a}{2} t
\]

\[
q(t) = \left( \frac{n + \frac{a}{2} t_1}{e^{r t_1} - 1} \right) (e^{r t} - 1) - \frac{a}{2} t \quad \text{where} \quad 0 \leq t \leq t_1
\]

The output function found is a good candidate for extremal, but further knowledge is needed to prove that it represents a minimum.

4. The quantity demanded of a firm \( q(t) \) depends on the own price of the good \( p(t) \) and the rate of change of the price with time \( p'(t) \) according to the function \( q(t) = ap(t) + bp'(t) \). On the other hand, the production costs of the firm are given by \( c(t) = mq^2 + nq + k \). If the price in the initial moment is \( p_0 \) and the price at time \( t_1 \) is \( p_1 \), find the optimal price function that maximizes the firm’s profits over the period \( 0 \leq t \leq t_1 \).

Solution:

The optimization problem is

\[
\max \int_0^{t_1} [p(t)q(t) - c(q)]dt
\]

subject to \( p(0) = p_0 \) and \( p(t_1) = p_1 \)

Solving for the objective function,
\[ t \int_0^t [p(t)q(t) - c(q)] \, dt = \int_0^t [p(ap + bp') - (mq^2 + nq + k)] \, dt = \]
\[ = \int_0^t [ap^2 + bpp' - m(ap + bp')^2 - n(ap + bp') - k] \, dt = \]
\[ = \int_0^t [ap^2 + bpp' - ma^2 p^2 - 2mabpp' - mb^2 (p')^2 - nap - nbp' - k] \, dt = \]
\[ = \int_0^t [a(1- ma)p^2 + (1-2ma)bpp' - mb^2 (p')^2 - nap - nbp' - k] \, dt \]

Thus, the integrand gives \( F[t, p(t), p'(t)] \) where the partial derivatives are

\[ F_p = 2a(1- ma)p + (1-2ma)bp' - na \quad \text{and} \quad F_{pp} = (1-2ma)bp - 2mb^2 p' - nb \]

Substituting in the Euler’s equation,

\[ 2a(1- ma)p + (1-2ma)bp' - na = \frac{d}{dt} \left[ (1-2ma)bp - 2mb^2 p' - nb \right] \]

\[ 2a(1- ma)p + (1-2ma)bp' - na = (1-2ma)bp' - 2mb^2 p^* \]

Rearranging,

\[ 2mb^2 p^* + 2a(1- ma)p = na \]

gives a second-order linear differential equation that could be solved by the well-known procedure. Normalizing the equation, we obtain

\[ p^* + \frac{a(1- ma)}{mb^2} p = \frac{na}{2mb^2} \]

\[ \begin{align*}
  a_1 &= 0 \\
  a_2 &= \frac{a(1- ma)}{mb^2} \\
  c &= \frac{na}{2mb^2}
\end{align*} \]

The particular integral is

\[ p_p = \frac{c}{a_2} = \frac{nam b^2}{2mb^2 a(1- ma)} = \frac{n}{2(1- ma)} \]

For the complementary function, we find the characteristic roots

\[ r_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} = \frac{0 \pm \sqrt{0 - 4a(1- ma) / mb^2}}{2} = \pm \sqrt{\frac{a(ma - 1)}{mb^2}} \]

Hence, the price function which optimizes the profit of the firm becomes

\[ p(t) = A_1 e^{r_1 t} + A_2 e^{r_2 t} + \frac{n}{2(1- ma)} \]

where \( r_1 = \sqrt{\frac{a(ma - 1)}{mb^2}} \) and \( r_2 = -\sqrt{\frac{a(ma - 1)}{mb^2}} \)

Since it could be expected that for the firm \( a < 0 \) (quantity demanded and price are negatively related) and \( m > 0 \) (costs increase rapidly with the level of output), then the term \( \sqrt{\frac{a(ma - 1)}{mb^2}} \) is
positive, which means that the price function has equal distinct real roots with opposite signs. Therefore, its time path is dynamically unstable. Assuming further that \( n > 0 \), the particular integral that gives the intertemporal equilibrium for the price takes a positive value and becomes meaningful. However, given that one characteristic root is positive, the time path of price is divergent and this intertemporal equilibrium cannot be achieved.

5. For the firm in the previous problem, assume the same demand and cost functions. However, assume further that the costs are incurred now, as output is produced in the present moment, but revenues are to be received in the future when an order is made. Thus, the present value of the future revenues of the firm depends on the rate of interest \( r \). Solve again the optimization problem for the proper price function of the firm that maximizes its profit considering the time factor in receiving revenues.

**Solution:**

Again, we have the demand function 

\[ q(t) = ap(t) + bp'(t) \]

and the cost function 

\[ c(t) = mq^2 + nq + k \]

over the interval \( 0 \leq t \leq t_1 \)

Discounting the future revenues, we get a slightly modified optimization problem

\[
\max \int_0^{t_1} \left[ p(t)q(t)e^{-rt} - c(q) \right] dt
\]

subject to \( p(0) = p_o \) and \( p(t_1) = p_1 \)

Substituting the respective functions,

\[
\int_0^{t_1} \left[ p(t)q(t)e^{-rt} - c(q) \right] dt = \int_0^{t_1} \left[ p(ap + bp')e^{-rt} - (mq^2 + nq + k) \right] dt =
\]

\[
= \int_0^{t_1} \left[ ap^2 e^{-rt} + bpp'e^{-rt} - m(ap + bp')^2 - n(ap + bp') - k \right] dt =
\]

\[
= \int_0^{t_1} \left[ ap^2 e^{-rt} + bpp'e^{-rt} - ma^2 p^2 - 2mabpp' - mb^2 (p')^2 - nap - nbp' - k \right] dt =
\]

\[
= \int_0^{t_1} \left[ a(e^{-rt} - ma)p^2 + (e^{-rt} - 2ma)bpp' - mb^2 (p')^2 - nap - nbp' - k \right] dt
\]

For the functional \( F[t, p(t), p'(t)] \), the partial derivatives are

\[ F_p = 2a(e^{-rt} - ma)p + (e^{-rt} - 2ma)bp' - na \quad \text{and} \quad F_{p'} = (e^{-rt} - 2ma)bp - 2mb^2 p' - nb \]

Using the Euler’s equation,

\[ 2a(e^{-rt} - ma)p + (e^{-rt} - 2ma)bp' - na = \frac{d}{dt} \left[ (e^{-rt} - 2ma)bp - 2mb^2 p' - nb \right] \]

\[ 2a(e^{-rt} - ma)p + (e^{-rt} - 2ma)bp' - na = (e^{-rt} - 2ma)bp' - r be^{-rt} p - 2mb^2 p' \]

Rearranging,

\[ 2mb^2 p'' + 2a(e^{-rt} - ma)p + r be^{-rt} p = na \]
2mb^2 p^* + \left[2a(e^{-rt} - ma) + rbe^{-rt}\right] p = na

Normalizing the equation

\[ p^* + \frac{2a(e^{-rt} - ma) + rbe^{-rt}}{2mb^2} p = \frac{na}{2mb^2} \]

where the parameters are

\[ a_1 = 0 \quad a_2 = \frac{2a(e^{-rt} - ma) + rbe^{-rt}}{2mb^2} \quad c = \frac{na}{2mb^2} \]

The particular integral is

\[ p_\pi = \frac{c}{a_2} = \frac{na2mb^2}{2mb^2 \left[2a(e^{-rt} - ma) + rbe^{-rt}\right]} = \frac{na}{2a(e^{-rt} - ma) + rbe^{-rt}} \]

For the complementary function we find

\[ r_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} = \frac{0 \pm \sqrt{0 - 4 \left[2a(e^{-rt} - ma) + rbe^{-rt}\right]}}{mb^2} = \pm \sqrt{\frac{2a(ma - e^{-rt}) - rbe^{-rt}}{mb^2}} \]

The final form of the price function that maximizes the firm’s profit is

\[ p(t) = A_1 e^{rt} + A_2 e^{st} + \frac{na}{2a(e^{-rt} - ma) + rbe^{-rt}} \]

where \( r_1 = \sqrt{\frac{2a(ma - e^{-rt}) - rbe^{-rt}}{mb^2}} \) and \( r_2 = -\sqrt{\frac{2a(ma - e^{-rt}) - rbe^{-rt}}{mb^2}} \)

Upon analyzing the particular integral, we conclude that the intertemporal equilibrium price is positive when \( a, b < 0 \) and \( m, n > 0 \). This implies, as can be expected from economic theory, that quantity demanded and own price are negatively related and costs increase rapidly with the level of output. However, the fact that \( b < 0 \) also means that consumers expect price to fall in the future as they observe it rise at present, so they would reduce current demand.

With these predicted parameters, we also obtain that the expression under the square root of each of the characteristic roots is positive. (Can you see why?) This means that the two characteristic roots are equal in value but have opposite signs. With one root positive the time path of price cannot be dynamically stable, which implies that the intertemporal equilibrium cannot be reached as time passes.

6. Consider a firm facing a simple demand function where the quantity demanded \( q(t) \) is negatively related to the own price of the good \( p(t) \) such that \( q(t) = ap(t) + b \) where \( a \) and \( b \) are parameters.

On the other hand, the firm’s costs include production costs \( m[q'(t)]^2 \) and inventory costs \( nq(t) \) where \( m \) and \( n \) are positive constants. The inventory accumulated by time \( t \) is \( q(t) \), while the rate of change of inventory with time is the production rate \( q'(t) \), such that \( mq'(t) \) is the per unit cost of production. If the price in the initial moment is \( p_o \) and the price at time \( t_1 \) is \( p_1 \), find the optimal price function that maximizes the firm’s profits over the period \( 0 \leq t \leq t_1 \).

Solution:

The total costs of the firm are
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\[ c(q) = m[q'(t)]^2 + nq(t) \]

Furthermore, from the demand function, we have \( q'(t) = ap'(t) \). Expressing the profit function of the firm,
\[ p(t)q(t) - c(t) = p(ap + b) - m[ap'(t)]^2 - n(ap + b) = ap^2 + bp - ma^2(p')^2 - nap - nb = \]
\[ = ap^2 + (b - na)p - ma^2(p')^2 - nb \]

Thus, the optimization problem becomes
\[ \max \int_0^{t_i} [p(t)q(t) - c(q)] dt \]
subject to \( p(0) = p_o \) and \( p(t_i) = p_i \)

Alternatively, the objective function is
\[ \int_0^{t_i} [p(t)q(t) - c(q)] dt = \int_0^{t_i} [ap^2 + (b - na)p - ma^2(p')^2 - nb] dt \]

The integrand gives \( F[t, p(t), p'(t)] \) where \( F_p = 2ap + b - na \) and \( F_{p'} = -2ma^2 p' \). From the Euler’s equation,
\[ 2ap + b - na = \frac{d}{dt} (-2ma^2 p') \]
\[ 2ap + b - na = -2ma^2 p^* \]
\[ 2ma^2 p^* + 2ap = na - b \] or
\[ p^* + \frac{1}{ma} p = \frac{na - b}{2ma^2} \]

The particular integral is
\[ p_p = \frac{(na - b)ma}{2ma^2} = \frac{na - b}{2a} \]

We see that with \( a < 0 \), the particular integral is meaningful only when \( b > 0 \). Furthermore,
\[ r_{1,2} = \frac{0 \pm \sqrt{0 - 4/ma}}{2} = \pm \sqrt{-ma} \]

Accounting for the fact that \( a < 0 \) and \( m > 0 \), we have a positive value under the square root and we have one positive and one negative characteristic root. Thus, the optimal price function is
\[ p(t) = A_1 e^{\sqrt{-ma}t} + A_2 e^{-\sqrt{-ma}t} + \frac{na - b}{2a} \]

Because of the presence of one positive root, again we exclude dynamic stability for the price function.

7. Consider the standard case of a firm operating in the short run with a production function \( q(K, L) \) where \( L = L(t) \). The price of labor is \( w \), the price of capital is \( r \), the price of the finished product of the firm is \( p \), and the rate of discount is \( \rho \). The firm maximizes the discounted present value of its profit obtainable infinitely. Use the techniques of the calculus of variations to prove that at the optimum the firm would pay for the variable input a price equal to the value of its marginal product.
What happens when the firm starts operating in the long run, that is, the production function changes to \( q[K(L), L] \)?

**Solution:**

We can express the short-term profit of the firm as

\[
\pi = pq(\bar{K}, L) - wL - r\bar{K}
\]

Thus, the optimization problem for the firm is

\[
\max \int_0^\infty \pi(t)e^{-\rho t} \, dt = \int_0^\infty \left[ pq(\bar{K}, L) - wL - r\bar{K} \right] e^{-\rho t} \, dt
\]

\[
F = \left[ pq(\bar{K}, L) - wL - r\bar{K} \right] e^{-\rho t}
\]

where by Euler’s equation

\[
F_L = (pq_L - w)e^{-\rho t}
\]

and

\[
F_L' = 0 \quad \text{and} \quad F_L = \frac{dF_L}{dt}
\]

Since \( F_L' \) is a constant function, we have \( F_L = \frac{dF_L}{dt} = 0 \) and

\[
F_L = (pq_L - w)e^{-\rho t} = 0
\]

Thus, at the optimum, we must have \( pq_L = w \) and

\[
VMP_L = pq_L = w
\]

or the firm would pay for labor a price (wage) exactly equal to the value of its marginal product. This result is consistent with what we get under static optimization. In a long-run situation, the profit can be expressed as

\[
\pi = pq[K(L), L] - wL - rK(L)
\]

and the new optimization problem is

\[
\max \int_0^\infty \pi(t)e^{-\rho t} \, dt = \int_0^\infty \left\{ pq[K(L), L] - wL - rK(L) \right\} e^{-\rho t} \, dt
\]

The integrand now is

\[
F = \left\{ pq[K(L), L] - wL - rK(L) \right\} e^{-\rho t}
\]

\[
F_L = \left[ p \left( q_L \frac{dK}{dL} + q_L \right) - w - r \frac{dK}{dL} \right] e^{-\rho t}
\]

and

\[
F_L = \frac{dF_L}{dt} = 0 \quad \text{and} \quad F_L' = 0
\]

Thus,

\[
F_L = \left[ p \left( q_L \frac{dK}{dL} + q_L \right) - w - r \frac{dK}{dL} \right] e^{-\rho t} = 0
\]

where the parenthesized expression must be zero for an optimum to hold.

\[
pL \left( q_L \frac{dK}{dL} + q_L \right) - w - r \frac{dK}{dL} = 0
\]

Transforming the left side,

\[
pq_L \frac{dK}{dL} + pq_L - w - r \frac{dK}{dL} = 0
\]


(pqK - r) \frac{dK}{dL} + pq_L - w = 0 \quad (dL)

and multiplying both sides by \( dL \),

(pqK - r)dK + (pq_L - w)dL = 0

Since the firm is operating in the long run, both labor and capital vary. Hence, we have nonzero values for the differentials \( dK \) and \( dL \). Therefore, the only possibility for the left-hand side expression to be equal to zero is for both parenthesized terms to be zeros as well, that is, at the optimum we must have

\[
\begin{align*}
pq_L - w &= 0 \\
pq_K - r &= 0
\end{align*}
\]

and consequently

\[
\begin{align*}
VMP_L &= pq_L = w \\
VMP_K &= pq_K = r
\end{align*}
\]

Even in the long run the firm would pay for inputs exactly as much as the values of their marginal products. This result confirms what we obtained previously when we discussed the optimal input decisions of the firm using the techniques of static optimization.

8. An industry represents a near monopoly. The total industry demand at time \( t \) is \( q(t) = a - bp(t) \), \((a, b > 0) \) where \( q(t) \) is the quantity demanded in the industry and \( p(t) \) is the price. The near monopoly is one large firm that sets the industry price and behaves monopolistically. A competitive fringe of small firms exists that behave competitively – they take the monopolist’s price as given. New small firms enter if the near monopolist charges a price higher than \( p^* \). Thus, the output of the fringe \( f(t) \) increases if \( p(t) > p^* \), and decreases otherwise. This results in the differential equation

\[
f''(t) = k\left[p(t) - p^*\right] \quad k > 0
\]

If the average cost of the near monopolist is \( c \) where \( p^* > c \) and the interest rate is \( r \), express the discounted present value of the profit of the large firm. Then maximize this present value, assuming \( f(t) \) to be the stock variable and \( p(t) \) the control variable.

**Solution:**

The present value of the profit function of the large firm can be expressed as

\[
\pi(t)e^{-rt} = \left[p(t) - c\right]\left[a - bp(t) - f(t)\right]e^{-rt}
\]

We have to maximize it. No endpoint conditions are given, so

\[
\max \int_0^\infty \pi(t)e^{-rt} dt = \int_0^\infty \left[p(t) - c\right]\left[a - bp(t) - f(t)\right]e^{-rt} dt
\]

Furthermore, from the differential equation \( f''(t) = k\left[p(t) - p^*\right] \), we have

\[
p(t) = \frac{f''(t)}{k} + p^*
\]

So, substituting for \( p(t) \) and simplifying,

\[
\int_0^\infty \left[\frac{f'}{k} + p^* - c\right]\left[a - b\left(\frac{f'}{k} + p^*\right) - f\right]e^{-rt} dt
\]

Thus, the functional in this dynamic optimization problem is

\[
F[t, f(t), f'(t)] = \left(\frac{f'}{k} + p^* - c\right)\left[a - b\left(\frac{f'}{k} + p^*\right) - f\right]e^{-rt} \quad \text{where}
\]
\[ F_j = -\left(\frac{f'}{k} + p^* - c\right)e^{-rt} \]

\[ F_j' = \frac{1}{k}\left[a - b\left(\frac{f'}{k} + p^*\right) - f\right]e^{-rt} - b\left(\frac{f'}{k} + p^* - c\right)e^{-rt} \]

and from Euler’s equation,

\[ -\left(\frac{f'}{k} + p^* - c\right)e^{-rt} = \frac{d}{dt}\left\{\frac{1}{k}\left[a - b\left(\frac{f'}{k} + p^*\right) - f\right]e^{-rt} - b\left(\frac{f'}{k} + p^* - c\right)e^{-rt}\right\} \]

\[ -\frac{f'}{k} + p^* + c = -\frac{b f'}{k^2} - \frac{f'}{k} - \frac{b f'}{k^2} \]

\[ \frac{2bf'}{k^2} = p^* - c \]

\[ f'' = \frac{(p^* - c)k^2}{2b} \]

This is a simple second-order differential equation, the solution of which is

\[ f(t) = \frac{(p^* - c)k^2t^2}{4b} + c_1t \quad \text{where} \]

\[ f'(t) = \frac{(p^* - c)k^2t}{2b} + c_1 \]

So, the industry price is

\[ p(t) = \frac{f'(t)}{k} + p^* = \frac{(p^* - c)kt}{2b} + \frac{c_1}{k} + p^* \]

and the output of the monopolist is

\[ a - bp(t) - f(t) = a - \frac{b(p^* - c)kt}{2b} - \frac{bc_1}{k} - bp^* - \frac{(p^* - c)k^2t^2}{4b} - c_1t = \]

\[ = a - \frac{c_1(b + kt)}{k} - \frac{(p^* - c)kt(2b + kt)}{4b} \]

9. The aggregate output of a county \( Y(K) \) is a function of the stock of capital accumulated in the nation \( K(t) \) at a particular moment in time. If the investment rate is the increase of the capital stock with time \( K'(t) \), use the simple Keynesian model to maximize the level of total utility \( U[C(t)] \) the nation achieves from its aggregate consumption. Assume that \( K(0) = K_o \) and \( K(T) = K_T \).

**Solution:**

According to the simple Keynesian model,

\[ C(t) = Y(K) - K'(t) \]

or

\[ C(t) = Y[K(t)] - K'(t) \]
Thus, the total utility function becomes

\[ U[C(t)] = U\{Y[K(t)] - K'(t)\} \]

and the optimization problem is

\[
\max \int_0^T U[C(t)] \, dt = \int_0^T U\{Y[K(t)] - K'(t)\} \, dt
\]

subject to \( K(0) = K_0 \) and \( K(T) = K_T \)

where the integrand is \( F[t, K(t), K'(t)] = U\{Y[K(t)] - K'(t)\} \). From \( U[C(t)] \) we also have

\[ U' = \frac{dU}{dC} \quad \text{and} \quad \frac{dC}{dK} = Y'[K(t)] \quad \text{and} \quad \frac{dC}{dK'} = -1 \]

Therefore, the partial derivatives are

\[ F_K = U'[C(t)]Y'[K(t)] \quad \text{and} \quad F_{K'} = -U'[C(t)] \]

Using the chain rule, we get

\[ U'[C(t)]Y'[K(t)] = \frac{d}{dt}\{-U'[C(t)]\} \]

\[ U'[C(t)]Y'[K(t)] = -U'[C(t)]C' \]

\[ U'[C(t)]Y'[K(t)] = -U'[C(t)].\{Y'[K(t)K'] - K'^*\} \]

With specific forms of the above functions, we can solve this second-order differential equation in \( K \) and find the \( K(t) \) function that maximizes the given extremal.

10. The flow of output of a county is given by the production function \( Y(K) \) which depends on the stock of capital \( K(t) \). Gross investment is the sum of net investment \( K'(t) \) and the replacement rate of capital \( \rho K \) where capital depreciates at a proportional rate \( \rho \) for \( 0 \leq \rho \leq 1 \). Express the consumption flow \( C(t) \) and the utility the nation achieves from that flow of consumption \( U[C(t)] \). Maximize the discounted stream of utility over the interval \( 0 \leq t \leq T \), assuming that the discount rate is \( r \).

**Solution:**

From the simple Keynesian model, we have

\[ C(t) = Y[K(t)] - I_g(t) \]

We know that gross investment is the sum of net investment and the rate of replacement, or

\[ I_g(t) = I_n + \rho K(t) = K'(t) + \rho K(t) \]

Thus, for the consumption function, we have

\[ C(t) = Y[K(t)] - I_g(t) = Y[K(t)] - K'(t) - \rho K(t) \]

And further for the utility function

\[ U[C(t)] = U\{Y[K(t)] - K'(t) - \rho K(t)\} \]

Thus, the optimization problem becomes
\[
\max_{0}^{T} \left[ U[C(t)]e^{-\nu t} dt = \int_{0}^{T} \left\{ Y[K(t)] - K'(t) - \rho K(t) \right\} e^{-\nu t} dt \right.
\]
subject to \( K(0) = K_0 \) and \( K(T) = K_T \).

with a functional \( F[t, K(t), K'(t)] = U \left\{ Y[K(t)] - K'(t) - \rho K(t) \right\} e^{-\nu t} \). We also have the derivatives
\[
U' = \frac{dU}{dC} \quad \text{and} \quad \frac{dC}{dK} = Y'[K(t)] - \rho \quad \text{and} \quad \frac{dC}{dK'} = -1. \]

Therefore,
\[
F_K = U'[C(t)] \left\{ Y'[K(t)] - \rho \right\} e^{-\nu t} = U'[Y[K(t)] - K'(t) - \rho K(t)] \left\{ Y'[K(t)] - \rho \right\} e^{-\nu t}
\]
and
\[
F_{K'} = -U'[C(t)] e^{-\nu t} = -U'[Y[K(t)] - K'(t) - \rho K(t)] \left\{ Y'[K(t)] - \rho \right\} e^{-\nu t}
\]

Using the chain rule, we get \( C' = \frac{dC}{dt} = Y'K' - K^n - \rho K' \). Substituting in the Euler’s equation,
\[
U'.(Y' - \rho)e^{-\nu t} = \frac{d}{dt}(-U'e^{-\nu t}) = -U'C'e^{-\nu t} + rU'e^{-\nu t}
\]
\[
U'.(Y' - \rho)e^{-\nu t} = -U'^n(Y'K' - K^n - \rho K')e^{-\nu t} + rU'e^{-\nu t}
\]

Cancelling the exponential term,
\[
U'.(Y' - \rho) = -U'^n(Y'K' - K^n - \rho K') + rU'
\]

We can solve this second-order differential equation if we are given some specific form of the utility function. Transforming the previously obtained equation,
\[
U'.(Y' - \rho)e^{-\nu t} = \frac{d}{dt}(-U'e^{-\nu t})
\]
\[
U'.(Y' - \rho)e^{-\nu t} = -\frac{dU'}{dt}e^{-\nu t} + rU'e^{-\nu t}
\]
\[
U'.(Y' - \rho) = -\frac{dU'}{dt} + rU'
\]
\[
\frac{dU'}{dt} = U'(r + \rho - Y')
\]
\[
\frac{dU'/dt}{U'} = r + \rho - Y'
\]

where the term \( \frac{dU'/dt}{U'} \) gives the marginal function over the total function of the marginal utility of consumption of the nation \( U' \), that is, the rate of growth of the marginal utility. With total utility maximized, we obtain that the rate of growth of marginal utility should be equal to the sum of the discount rate and the depreciation rate less the marginal product of capital \( Y'(K) = \frac{dY}{dK} = Y_K \). The rate of growth of marginal utility is often referred to in economic theory as capital gains. We get that the optimal time path requires capital gains to be exactly equal to the term on the right. If capital gains exceed the sum of the discount rate and the depreciation rate less the marginal product of capital, then the marginal product of capital is too high and there are high returns to capital. Therefore, more capital should be applied, and, as a result of its increase, consumption would increase too. In the opposite case, when capital gains are too small, there is too much capital applied and, hence, capital accumulation and consumption should be decreased.
11. Maximize the discounted stream of utility from consumption of a nation that has a linear production function 
\(Y[K(t)] = \alpha K(t)\) where \(\alpha > 0\), if the utility from the flow of consumption takes the form 
\(U[C(t)] = \ln C(t)\) over the interval \(0 \leq t \leq T\). Gross investment is the sum of net investment \(K'(t)\) and linear depreciation and is given by the equation

\[I_g(t) = K'(t) + \beta + \rho K(t)\]

\(0 \leq \rho \leq 1\) \(\beta > 0\)

Assume a discount rate of \(r\).

**Solution:**

For the flow of consumption, we have

\[C(t) = Y[K(t)] - I_g(t) = \alpha K(t) - K'(t) - \beta - \rho K(t)\]

We have the optimization problem

\[
\max_{0 \leq t \leq T} \int_0^T U[C(t)]e^{-rt} dt = \int_0^T \ln[\alpha K(t) - K'(t) - \beta - \rho K(t)]e^{-rt} dt
\]

subject to \(K(0) = K_0\) and \(K(T) = K_T\)

where the functional is simply \(F[t, K(t), K'(t)] = \ln(\alpha K - K' - \beta - \rho K)e^{-rt}\). We also need some essential derivatives:

\[
U' = \frac{dU}{dC} = \frac{1}{\alpha K - K' - \beta - \rho K}n
Y' = \frac{dY}{dK} = \alpha
\]

\[
\frac{dC}{dK} = \alpha - \rho \quad \text{and} \quad \frac{dC}{dK'} = -1
\]

Therefore,

\[
F_K = \frac{(\alpha - \rho)}{\alpha K - K' - \beta - \rho K}e^{-rt} \quad \text{and} \quad F_{K'} = -\frac{1}{\alpha K - K' - \beta - \rho K}e^{-rt}
\]

Using the Euler’s equation,

\[
\frac{(\alpha - \rho)}{\alpha K - K' - \beta - \rho K}e^{-rt} = \frac{d}{dt}\left(-\frac{1}{\alpha K - K' - \beta - \rho K}e^{-rt}\right)
\]

\[
\frac{(\alpha - \rho)}{\alpha K - K' - \beta - \rho K}e^{-rt} = \frac{(\alpha K' - K' - \rho K')}{(\alpha K - K' - \beta - \rho K)}e^{-rt} + \frac{r}{\alpha K - K' - \beta - \rho K}e^{-rt}
\]

\[
\frac{(\alpha - \rho)}{\alpha K - K' - \beta - \rho K} = \frac{(\alpha K' - K'' - \rho K')}{(\alpha K - K' - \beta - \rho K)} + \frac{r}{\alpha K - K' - \beta - \rho K}
\]

\[
\alpha - \rho - r = \frac{\alpha K' - K'' - \rho K'}{\alpha K - K' - \beta - \rho K}
\]

First, we can easily check that this is equivalent to the result

\[
\frac{dU'}{dt} = r + \rho - Y'
\]
so again, at the optimum, capital gains should equal the sum of the discount rate and the depreciation rate minus the marginal product of capital, here equal to $\alpha$. Cross-multiplying

\[(\alpha - \rho - r)(\alpha - \rho)K - (\alpha - \rho - r)K' - (\alpha - \rho - r)\beta = (\alpha - \rho)K' - K^*\]

\[K^* - (2\alpha - 2\rho - r)K' + (\alpha - \rho - r)(\alpha - \rho)K = (\alpha - \rho - r)\beta\]

which is a linear second-order differential equation in $K(t)$. Therefore, the particular integral giving the intertemporal equilibrium value of the capital stock is

\[K_p = \frac{\beta}{\alpha - \rho}\]

Equilibrium capital stock will only make sense if $\alpha > \rho$, since by definition $\beta > 0$. To solve for the complementary function it is convenient to set $\alpha - \rho = m$. Then the equation becomes

\[K^* - (2m - r)K' + (m^2 - mr)K = (m - r)\beta\]

The characteristic roots are

\[r_{1,2} = \frac{2m - r \pm \sqrt{(2m - r)^2 - 4(m^2 - mr)}}{2} = \frac{2m - r \pm \sqrt{4m^2 - 4mr + r^2 - 4m^2 + 4mr}}{2} = \frac{2m - r \pm \sqrt{r^2}}{2} = \frac{2m - r \pm r}{2}\]

Hence, $r_1 = m$ and $r_2 = m - r$. Since for a positive intertemporal equilibrium we assumed $\alpha > \rho$, which means $m > 0$, we have at least one root positive. The optimal time path of the capital function that would maximize the nation’s total utility is, therefore,

\[K(t) = A_1e^{mt} + A_2e^{(m-r)t} + \frac{\beta}{m}\]

In view of the positive characteristic root, we conclude that the time path of capital is divergent. Using the endpoint conditions that $K(0) = K_o$ and $K(T) = K_T$, we can find the arbitrary constants $A_1$ and $A_2$.

\[K(0) = A_1 + A_2 + \frac{\beta}{m} = K_o \quad \text{or} \quad A_2 = K_o - \frac{\beta}{m} - A_1\]

\[K(T) = A_1e^{mT} + A_2e^{(m-r)T} + \frac{\beta}{m} = K_T\]

\[A_1e^{mT} + \left(K_o - A_1 - \frac{\beta}{m}\right)e^{(m-r)T} + \frac{\beta}{m} = K_T\]

\[A_1e^{mT} + \left(K_o - \frac{\beta}{m}\right)e^{(m-r)T} - A_2e^{(m-r)T} = K_T - \frac{\beta}{m}\]

\[A_1\left[e^{mT} - e^{(m-r)T}\right] = K_T - \frac{\beta}{m} - \left(K_o - \frac{\beta}{m}\right)e^{(m-r)T}\]

\[A_1 = \frac{K_T - \beta - \left(K_o - \frac{\beta}{m}\right)e^{(m-r)T}}{e^{mT} - e^{(m-r)T}}\]

\[A_2 = K_o - \frac{\beta}{m} - A_1 = K_o - \frac{\beta}{m} - \frac{K_T - \beta - \left(K_o - \frac{\beta}{m}\right)e^{(m-r)T}}{e^{mT} - e^{(m-r)T}}\]
12. For the nation described in the previous problem, assume everything is the same except the utility from the flow of consumption is given by the function

\[ U[C(t)] = [C(t)]^n \text{ where } 0 \leq n \leq 1. \]

**Solution:**

Again, for the flow of consumption, we have

\[ C(t) = Y[K(t)] - I_g(t) = \alpha K(t) - K'(t) - \beta - \rho K(t) \]

We optimize

\[
\max \int_0^T U[C(t)] e^{-rt} dt = \int_0^T \left[ \alpha K(t) - K'(t) - \beta - \rho K(t) \right]^n e^{-rt} dt
\]

subject to

\[ K(0) = K_o \quad \text{and} \quad K(T) = K_T \]

The functional is

\[ F\left[t, K(t), K'(t)\right] = (\alpha K - K' - \beta - \rho K)^n e^{-rt} \]

Therefore,

\[ F_K = n(\alpha - \rho)(\alpha K - K' - \beta - \rho K)^{n-1} e^{-rt} \quad F_{K'} = -n(\alpha K - K' - \beta - \rho K)^{n-1} e^{-rt} \]

Using the Euler’s equation,

\[ n(\alpha - \rho)(\alpha K - K' - \beta - \rho K)^{n-1} e^{-rt} = \frac{d}{dt} \left[ -n(\alpha K - K' - \beta - \rho K)^{n-1} e^{-rt} \right] \]

To differentiate the right-hand side, we need to apply the product rule.

\[ n(\alpha - \rho)(\alpha K - K' - \beta - \rho K)^{n-1} e^{-rt} = -n(n-1)(\alpha K - K' - \beta - \rho K)^{n-2}(\alpha K' - K'' - \rho K')e^{-rt} + \]

\[ +nr(\alpha K - K' - \beta - \rho K)^{n-1} e^{-rt} \]

and cancelling out the common terms on the left and on the right,

\[ \alpha - \rho = -(n-1)(\alpha K - K' - \beta - \rho K)^{-1}(\alpha K' - K'' - \rho K') + r \]

\[ \frac{(1-n)(\alpha K' - K'' - \rho K')}{\alpha K - K' - \beta - \rho K} = \alpha - \rho - r \]

For simplicity, we can set \( m = \alpha - \rho \), which gives

\[ \frac{(1-n)(mK' - K'')}{mK - K' - \beta} = m - r \]

Cross-multiplying and rearranging,

\[ (1-n)mK' - (1-n)K'' = (m-r)mK - (m-r)K' - \beta(m-r) \]

\[ (1-n)K'' - (2m - mn - r)K' + (m-r)mK = \beta(m-r) \]

Normalizing the equation,
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\[ \frac{K^* + (mn + r - 2m)}{1 - n} \frac{K'}{1 - n} + \frac{(m^2 - mr)}{1 - n} K = \frac{\beta(m - r)}{1 - n} \]

where the parameters are

\[ a_1 = \frac{mn + r - 2m}{1 - n}, \quad a_2 = \frac{m^2 - mr}{1 - n}, \quad \text{and} \quad c = \frac{\beta(m - r)}{1 - n} \]

The particular integral is

\[ K_p = \frac{\beta(m - r)(1 - n)}{(1 - n)m(m - r)} = \frac{\beta}{\alpha - \rho} \]

Again, this intertemporal equilibrium value for the capital stock will be meaningful only if \( \alpha > \rho \), that is, if the marginal product of capital exceeds the rate of depreciation or the output produced by adding one more machine exceeds the rate at which this machine wears out. For the characteristic roots, we substitute the parameters in the well-known formula. We omit some of the more tedious arithmetic computations here.

\[
\begin{align*}
\rho_{1,2} &= -a_1 \pm \sqrt{a_1^2 - 4a_2} = \frac{-\frac{mn + r - 2m}{1 - n} \pm \sqrt{\left(\frac{mn + r - 2m}{1 - n}\right)^2 - \frac{4(m^2 - rm)}{(1 - n)}}}{2} \\
&= \frac{-\frac{mn + r - 2m}{1 - n} \pm \sqrt{\frac{(mn + r - 2m)^2 - (4m^2 - 4rm)(1 - n)}{(1 - n)^2}}}{2} \\
&= \frac{-\frac{mn + r - 2m}{1 - n} \pm \sqrt{-mn - r + 2m \pm \frac{(mn - r)}{2(1 - n)}}}{2(1 - n)} \\
&= -mn - r + 2m \pm \frac{mn - r}{2(1 - n)}
\end{align*}
\]

Thus, this long process of computing yields two simple roots

\[ \rho_1 = \frac{-mn - r + 2m - mn + r}{2(1 - n)} = \frac{2m(1 - n)}{2(1 - n)} = m \quad \text{and} \quad \rho_2 = \frac{-mn - r + 2m + mn - r}{2(1 - n)} = \frac{m - r}{1 - n} \]

Hence, the optimum time path for the capital function is

\[ K(t) = A_1 e^{\rho_1 t} + A_2 e^{\rho_2 t} + \frac{\beta}{m} = A_1 e^{\rho_1 t} + A_2 e^{\frac{(m - r)t}{1 - n}} + \frac{\beta}{m} = A_1 e^{(\alpha - \rho)t} + A_2 e^{\frac{(\alpha - \rho - r)t}{1 - n}} + \frac{\beta}{\alpha - \rho} \]

Since at least one characteristic root (\( \rho \)) is positive, the time path of capital is dynamically unstable. Using the endpoint conditions that \( K(0) = K_o \) and \( K(T) = K_T \), we can find the arbitrary constants \( A_1 \) and \( A_2 \).

\[ K(0) = A_1 + A_2 + \frac{\beta}{m} = K_o \quad \text{or} \quad A_2 = K_o - \frac{\beta}{m} - A_1 \]

\[ K(T) = A_1 e^{\rho_1 T} + A_2 e^{\frac{(m - r)T}{1 - n}} + \frac{\beta}{m} = K_T \]

\[ A_1 e^{\rho_1 T} + \left(K_o - A_1 - \frac{\beta}{m}\right) e^{\frac{(m - r)T}{1 - n}} + \frac{\beta}{m} = K_T \]
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\[ A e^{mT} + \left( K_o - \frac{\beta}{m} \right) e^{\frac{(m-r)T}{1-n}} - A e^{\frac{(m-r)T}{1-n}} = K_T - \frac{\beta}{m} \]

\[ A_1 \left[ e^{mT} - e^{\frac{(m-r)T}{1-n}} \right] = K_T - \frac{\beta}{m} - \left( K_o - \frac{\beta}{m} \right) e^{\frac{(m-r)T}{1-n}} \]

\[ A_2 = K_o - \frac{\beta}{m} - A_1 = K_o - \frac{\beta}{m} - \frac{K_T - \frac{\beta}{m} - \left( K_o - \frac{\beta}{m} \right) e^{\frac{(m-r)T}{1-n}}}{e^{mT} - e^{\frac{(m-r)T}{1-n}}} = \]

\[ \frac{\left( K_o - \frac{\beta}{m} \right) e^{mT} - \left( K_o - \frac{\beta}{m} \right) e^{\frac{(m-r)T}{1-n}} - K_T + \frac{\beta}{m} + \left( K_o - \frac{\beta}{m} \right) e^{\frac{(m-r)T}{1-n}}}{e^{mT} - e^{\frac{(m-r)T}{1-n}}} \]

Thus, the time path of capital is fully definitized.

13. Maximize

\[ \int_0^T U[C(t)]e^{-rt} dt \]

of a country that has the utility function \( U[C(t)] = [C(t)]^{0.5} \) if the discount rate is \( r = 0.09 \), the aggregate production function is \( Y[K(t)] = 0.15K \), and the investment function is \( I(t) = K'(t) + 20 + 0.05K(t) \). Assume boundary conditions \( K(0) = 240 \) and \( K(5) = 500 \).

Solution:

We can solve by following the well-known steps of maximization and the Euler’s equation, but a faster way to solve would be by direct substitution in the result obtained previously. We know that in its final form the differential equation in capital is

\[ K^* + \frac{(mn + r - 2m)}{1-n} K' + \frac{(m^2 - mr)}{1-n} K = \frac{\beta (m-r)}{1-n} \]

where the parameters are, respectively,

\[ r = 0.09 \quad n = 0.5 \quad \alpha = 0.15 \quad \rho = 0.05 \quad \beta = 20 \]

Therefore, we have \( m = \alpha - \rho = 0.15 - 0.05 = 0.1 \). Substituting in this equation,

\[ K^* + \frac{[0.1(0.5) + 0.09 - 2(0.1)]}{1-0.5} K' + \frac{[(0.1)^2 - 0.1(0.09)]}{1-0.5} K = \frac{20(0.1 - 0.09)}{1-0.5} \]
$$K'' + \frac{0.05 + 0.09 - 0.2}{0.5} K' + \frac{(0.01 - 0.009)}{0.5} K = 20(0.01)$$

$$K'' - 0.12K' + 0.002K = 0.4$$

$$K_0 = \frac{0.4}{0.002} = 200$$

$$r_{1,2} = 0.12 \pm \sqrt{(-0.12)^2 - 4(0.002)} = 0.12 \pm \sqrt{0.0064} = 0.12 \pm 0.08$$

$$r_1 = 0.1$$ and $$r_2 = 0.02$$

Hence, $$K(t) = A_1 e^{0.1t} + A_2 e^{0.02t} + 200$$

Since both characteristic roots are positive, the time path of capital diverges from the equilibrium value of 200 with the passage of time. To find the arbitrary constants $$A_1$$ and $$A_2$$, we use the boundary conditions $$K(0) = 240$$ and $$K(5) = 500$$.

$$K(0) = A_1 + A_2 + 200 = 240$$ or $$A_2 = 40 - A_1$$

$$K(5) = A_1 e^{0.1(5)} + A_2 e^{0.02(5)} + 200 = 500$$

$$A_1 e^{0.5} + A_2 e^{0.1} = 300$$

$$A_1 e^{0.5} + (40 - A_1)e^{0.1} = 300$$

$$A_1(e^{0.5} - e^{0.1}) = 300 - 40e^{0.1}$$

$$A_1 = \frac{300 - 40e^{0.1}}{e^{0.5} - e^{0.1}} = \frac{255.79}{0.54355} \approx 470$$

$$A_2 = 40 - 470 = -430$$

Therefore, the final form of the optimal capital function is

$$K(t) = 470e^{0.1t} - 430e^{0.02t} + 200$$

14. For the country presented in the previous example, assume that all the functions and parameters are the same but the utility function is logarithmic of the type $$U[C(t)] = \ln C(t)$$.

Solution:

The problem is again

$$\max T \int_0^T U[C(t)]e^{-rt}dt$$

with parameters

$$r = 0.09 \quad \beta = 20 \quad \alpha = 0.15 \quad \rho = 0.05 \quad m = \alpha - \rho = 0.15 - 0.05 = 0.1$$

With the logarithmic utility function, we previously obtained the second-order differential equation:

$$K'' - (2m - r)K' + (m^2 - mr)K = (m - r)\beta$$

Substituting in it,

$$K'' - [2(0.1) - 0.09]K' + [(0.1)^2 - 0.1(0.09)]K = (0.1 - 0.09)20$$
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\[ K'' - (0.2 - 0.09)K' + (0.01 - 0.009)K = 20(0.01) \]

\[ K'' - 0.11K' + 0.001K = 0.2 \]

\[ K_p = \frac{0.2}{0.001} = 200 \]

\[ r_{1,2} = \frac{0.11 \pm \sqrt{(-0.11)^2 - 4(0.001)}}{2} = \frac{0.11 \pm \sqrt{0.0081}}{2} = \frac{0.11 \pm 0.09}{2} \]

\[ r_1 = 0.1 \text{ and } r_2 = 0.01 \]

Hence, \[ K(t) = A_1e^{0.1t} + A_2e^{0.01t} + 200 \]

For the arbitrary constants \( A_1 \) and \( A_2 \),

\[ K(0) = A_1 + A_2 + 200 = 240 \quad \text{or} \quad A_2 = 40 - A_1 \]

\[ K(5) = A_1e^{0.1(5)} + A_2e^{0.01(5)} + 200 = 500 \]

\[ A_1e^{0.5} + A_2e^{0.05} = 300 \]

\[ A_1e^{0.5} + (40 - A_1)e^{0.05} = 300 \]

\[ A_1(e^{0.5} - e^{0.05}) = 300 - 40e^{0.05} \]

\[ A_1 = \frac{300 - 40e^{0.05}}{e^{0.5} - e^{0.05}} = \frac{257.949}{0.5974} \approx 432 \quad A_2 = 40 - 432 = -392 \]

Therefore, the final form of the optimal capital function is

\[ K(t) = 432e^{0.1t} - 392e^{0.01t} + 200 \]

15. For a country with a logarithmic utility function of the type \( U[C(t)] = \ln C(t) \), find the optimal time path of the capital function assuming the following parameters:

\[ r = 0.08 \quad \beta = 50 \quad \alpha = 0.25 \quad \rho = 0.05 \]

The boundary conditions are given to be \( K(0) = 280 \) and \( K(5) = 430 \). How does the time path of capital change, if the rate of depreciation \( \rho \) increases from 5% to 10%?

Solution:

For \( m \), we have \( m = \alpha - \rho = 0.25 - 0.05 = 0.2 \). The problem is again

\[ \max_0^\tau U[C(t)]e^{-\rho}dt \]

Again, we use the previously obtained second-order differential equation:

\[ K'' - (2m - r)K' + (m^2 - mr)K = (m - r)\beta \]

Substituting the parameters,

\[ K'' - [2(0.2) - 0.08]K' + [(0.2)^2 - 0.2(0.08)]K = (0.2 - 0.08)50 \]

\[ K'' - (0.4 - 0.08)K' + (0.04 - 0.016)K = 50(0.12) \]
\[ K'' - 0.32K' + 0.024K = 6 \]

\[ K_p = \frac{6}{0.024} = 250 \]

\[ r_{1,2} = \frac{0.32 \pm \sqrt{(-0.32)^2 - 4(0.024)}}{2} = \frac{0.32 \pm \sqrt{0.0064}}{2} = \frac{0.32 \pm 0.08}{2} \]

\[ r_1 = 0.2 \quad \text{and} \quad r_2 = 0.12 \]

Hence, \[ K(t) = A_1 e^{0.2t} + A_2 e^{0.12t} + 250 \]

For the arbitrary constants \( A_1 \) and \( A_2 \),

\[ K(0) = A_1 + A_2 + 250 = 280 \quad \text{or} \quad A_2 = 30 - A_1 \]

\[ K(5) = A_1 e^{0.2(5)} + A_2 e^{0.12(5)} + 250 = 430 \]

\[ A_1 e + A_2 e^{0.6} = 180 \]

\[ A_1 e + (30 - A_1) e^{0.6} = 180 \]

\[ A_1 (e - e^{0.6}) = 180 - 30e^{0.6} \]

\[ A_1 = \frac{180 - 30e^{0.6}}{e - e^{0.6}} \approx 140 \quad A_2 = 30 - 140 = -110 \]

Thus, \[ K(t) = 140e^{0.2t} - 110e^{0.12t} + 250 \]

If the proportional rate of depreciation is increased to 10%\( \rho = 0.1 \). Then, \[ m = 0.25 - 0.1 = 0.15 \]

Substituting again,

\[ K'' - [2(0.15) - 0.08]K' + [(0.15)^2 - 0.15(0.08)]K = (0.15 - 0.08)50 \]

\[ K'' - (0.3 - 0.08)K' + (0.0225 - 0.012)K = 50(0.07) \]

\[ K'' - 0.22K' + 0.0105K = 50 \]

\[ K_p = \frac{3.5}{0.0105} = \frac{1,000}{3} \]

\[ r_{1,2} = \frac{0.22 \pm \sqrt{(-0.22)^2 - 4(0.0105)}}{2} = \frac{0.22 \pm \sqrt{0.0064}}{2} = \frac{0.22 \pm 0.08}{2} \]

\[ r_1 = 0.15 \quad \text{and} \quad r_2 = 0.07 \]

Hence, \[ K(t) = A_1 e^{0.15t} + A_2 e^{0.07t} + \frac{1,000}{3} \]

For the arbitrary constants \( A_1 \) and \( A_2 \),

\[ K(0) = A_1 + A_2 + \frac{1,000}{3} = 280 \quad \text{or} \quad A_2 = \frac{160}{3} - A_1 \]
Chapter 12. Introduction to Dynamic Optimization: The Calculus of Variations

\[ K(5) = A_1e^{0.15(5)} + A_2e^{0.07(5)} + \frac{1000}{3} = 430 \]

\[ A_1e^{0.75} + A_2e^{0.35} = \frac{290}{3} \]

\[ A_1e^{0.75} - \left(\frac{160}{3} + A_1\right)e^{0.35} = \frac{290}{3} \]

\[ A_1(e^{0.75} - e^{0.35}) = \frac{290}{3} + \frac{160}{3}e^{0.35} \]

\[ A_1 = \frac{10(29 + 16e^{0.35})}{3(e^{0.75} - e^{0.35})} \approx 247 \]

\[ A_2 = -\frac{160}{3} - 247 \approx -300 \]

\[ K(t) = 247e^{0.15t} - 300e^{0.07t} + \frac{1000}{3} \]

One easily detectable effect of the increase in the depreciation rate is the increase of the intertemporal equilibrium value of capital. When the capital stock depreciates faster, more capital is necessary to sustain the economy in equilibrium. Because the characteristic roots are all positive, the time path of capital in both cases is divergent. However, with a greater depreciation, the divergence seems to be slower and take longer.

16. The utility a nation extracts from its aggregate consumption \( C(t) \) is \( U[C(t)] = \frac{[C(t)]^{1-\eta}}{1-\eta} \) so that the marginal utility of aggregate consumption is \( U' = C^{-\eta} \). The size of the population is given by the equation \( L(t) = L_0e^{rt} \). The investment function of the country is \( K' = Y(K, mL) - C - \rho K \) where \( m \) is the share of the labor force in the population and \( \rho \) is the share of capital that depreciates. The rate of discount is known to be \( r \). Maximize the discounted utility function of the nation from the initial moment to infinity. Analyze the growth rate of aggregate consumption. Write the differential equation for \( K(t) \).

**Solution:**

For aggregate consumption,

\[ C = Y(K, mL) - \rho K - K' \]

\[ \max_{0}^{\infty} U[C(t)]e^{-r\eta}dt = \int_{0}^{\infty} \frac{[Y(K, mL) - \rho K - K']^{1-\eta}}{1-\eta}e^{-r\eta}dt \]

\[ F(t, K, K') = \frac{[Y(K, mL) - \rho K - K']^{1-\eta}}{1-\eta}e^{-r\eta} \]

\[ F_K = \frac{[Y(K, mL) - \rho K - K']^{1-\eta}}{1-\eta}(Y_K - \rho)e^{-r\eta} \]

\[ F_{K'} = -\left[Y(K, mL) - \rho K - K'\right]^{1-\eta}e^{-r\eta} \quad \text{and using} \quad F_K = \frac{dF_K}{dt} \]

\[ [Y(K, mL) - \rho K - K']^{1-\eta}(Y_K - \rho)e^{-r\eta} = \frac{d}{dt}\left\{-\left[Y(K, mL) - \rho K - K'\right]^{1-\eta}e^{-r\eta}\right\} \]

\[ [Y(K, mL) - \rho K - K']^{1-\eta}(Y_K - \rho)e^{-r\eta} = \eta[Y(K, mL) - \rho K - K']^{-\eta-1}(Y_K' + mY_L - \rho K' - K)e^{-r\eta} + r[Y(K, mL) - \rho K - K']^{-\eta}e^{-r\eta} \]
\[ Y_K - \rho = \frac{\eta(Y_kK' + mY_tL' - \rho K' - K'')}{Y(K, mL) - \rho K - K'} + r \quad \text{where} \quad L' = \lambda L_o e^{\lambda t} \]

\[ \frac{Y_kK' + mY_t\lambda L_o e^{\lambda t} - \rho K' - K''}{Y(K, mL) - \rho K - K'} = \frac{Y_K - \rho - r}{\eta} \]

The left-hand side represents the rate of growth of aggregate consumption or

\[ \frac{C'}{C} = \frac{Y_K - \rho - r}{\eta} \]

The rate of growth of aggregate consumption is positively related to the marginal product of capital \(Y_K\). Furthermore, it is adversely related to both the interest rate and the rate of depreciation. As capital depreciates faster, it has to be replaced faster, which slows down the growth of consumption. A high interest rate would stimulate the population to substitute present consumption with future one for the sake of savings, thus again slowing the growth of consumption. Finally, the intertemporal elasticity of substitution \(\frac{1}{\eta}\) also has a positive effect on the increase in consumption. Again, it justifies the substitution of present consumption with future consumption. We can set \(a = Y_K - \rho\):

\[ \frac{aK' + mY_t\lambda L_o e^{\lambda t} - K''}{Y(K, mL) - \rho K - K'} = \frac{a-r}{\eta} \]

\[ \eta(aK' + mY_t\lambda L_o e^{\lambda t} - K'') = (a-r)(Y-K'-K') \]

\[ a\etaK' + \lambda m\eta Y_t L_o e^{\lambda t} - \eta K'' = (a-r)Y - (a-r)\rho K - (a-r)K' \]

\[ \eta K'' - (a\eta + a-r)K' - (a-r)\rho K = \lambda m\eta Y_t L_o e^{\lambda t} - (a-r)Y \]

\[ K'' - \frac{(a\eta + a-r)}{\eta} K' - \frac{(a-r)\rho}{\eta} K = \frac{\lambda m\eta Y_t L_o e^{\lambda t} - (a-r)Y}{\eta} \]

Notice that this second-order differential equation in \(K\) is one with a variable term on the right involving functions of time \(t\).

17. Referring to the previous problem, assume now that the optimization problem is to maximize the utility of an individual member of society from per capita consumption \(c(t)\) where the individual’s utility function is \(U[c(t)] = \left[\frac{c(t)}{1-\eta}\right]^{1-\eta}\). Everything else is the same. Maximize the discounted utility of the individual. Write the differential equation for \(K(t)\).

**Solution:**

From the investment equation

\[ K' = Y(K, mL) - Lc - \rho K \]

\[ c = \frac{Y(K, mL) - \rho K - K'}{L} \]

\[ \max_0^\infty U[c(t)] e^{-\eta t} dt = \int_0^\infty \frac{1}{1-\eta} \left[ \frac{Y(K, mL) - \rho K - K'}{L} \right]^{1-\eta} e^{-\eta t} dt \]
The rate of growth of aggregate consumption is positively related to the marginal product of capital \( Y_K \). Interest rate and the rate of depreciation affect the growth of aggregate consumption negatively.

We also see that the higher the growth rate of population \( \lambda \), the higher the growth of consumption (for \( \eta > 1 \)). Developing the expression further,

\[
\frac{C'}{C} = \frac{Y_K - \rho - r + (\eta - 1)\lambda}{\eta}
\]

Since \( C = LC \), we have \( \frac{C'}{C} = \frac{c'}{c} + \frac{L'}{L} \) or \( r_c = r_c + r_L \). Therefore, the growth rate of aggregate consumption is the sum of the rates of growth of per capita consumption and of population. Since \( r_L = \lambda \), we have for the growth rate of per capita consumption

\[
\frac{c'}{c} = \frac{Y_K - \rho - r - \lambda}{\eta}
\]

Like aggregate consumption, per capita consumption is positively related to the marginal product of capital and negatively to interest rate and depreciation. However, it is negatively affected by the growth of the population. As the size of the population increases, the individual’s consumption grows more slowly. We can set \( a = Y_K - \rho \):

\[
\frac{aK' + \lambda mY_f L_o e^{\lambda t} - K^*}{Y - \rho K - K'} = \frac{a - r + (\eta - 1)\lambda}{\eta}
\]

\[
\eta(aK' + mY_f \lambda L_o e^{\lambda t} - K^*) = [a - r + (\eta - 1)\lambda](Y - \rho K - K')
\]
\[ a\eta K' + \lambda \eta Y L e^{\alpha t} - \eta K^* = \left[ a - r + (\eta - 1)\lambda \right] Y - \left[ a - r + (\eta - 1) \right] \rho K - \left[ a - r + (\eta - 1) \right] K' \]

\[ \eta K^* - \left[ a\eta + a - r + (\eta - 1) \right] K' - \left[ a - r + (\eta - 1) \right] \rho K = \lambda \eta Y L e^{\alpha t} - \left[ a - r + (\eta - 1) \right] Y \]

\[ K^* - \left[ a\eta + a - r + (\eta - 1) \right] K' - \left[ a - r + (\eta - 1) \right] \rho K = \frac{\lambda \eta Y L e^{\alpha t}}{\eta} - \left[ a - r + (\eta - 1) \right] Y \]

Again we obtain a second-order differential equation in \( K \) where the variable term involves functions of time \( t \).

18. A worker’s life span is assumed to be \( T \) over which he obtains a fixed wage rate \( w \). He will receive a constant interest rate \( \rho \) on his accumulated life savings \( S(t) \) or pay the same rate on his accumulated debts (that will represent negative savings). If this worker’s consumption flow is \( C(t) \), express his capital (savings) accumulation. Suppose the worker does not have any initial endowment or inheritance, so the boundary conditions are \( S(0) = S(T) = 0 \). If his instantaneous utility function is \( U[C(t)] = \ln C(t) \) and it is discounted at the utility discount rate \( r \), maximize the discounted value of the utility function of the worker. What happens when, over the course of time, the utility discount rate and the interest rate on savings equalize?

**Solution:**

Here the worker’s flow income is his wage plus the increase in his savings, that is, \( w + \rho S(t) \). Thus, when he has positive savings, his flow income increases. With negative savings (debt), his total flow of income declines. Furthermore, the worker’s capital (savings) accumulation can be expressed as the difference between his flow income and his spending on consumption, that is,

\[ S'(t) = w + \rho S(t) - C(t) \]

Therefore, his consumption function is

\[ C(t) = w + \rho S(t) - S'(t) \]

We want to optimize

\[ \max_0^T U[C(t)]e^{-\alpha t} dt = \int_0^T \ln[w + \rho S(t) - S'(t)]e^{-\alpha t} dt \]

where \( S(0) = S(T) = 0 \).

We have \( F[t, S(t), S'(t)] = \ln(w + \rho S - S')e^{-\alpha t} \).

\[ F_S = \frac{\rho}{w + \rho S - S'}e^{-\alpha t} \quad \text{and} \quad F'_S = \frac{1}{w + \rho S - S'}e^{-\alpha t} \]

Using the Euler’s equation,

\[ \frac{\rho}{w + \rho S - S'}e^{-\alpha t} = \frac{d}{dt} \left[ \frac{1}{w + \rho S - S'}e^{-\alpha t} \right] \]

\[ \frac{\rho}{w + \rho S - S'}e^{-\alpha t} = \frac{(\rho S' - S'')(w + \rho S - S')e^{-\alpha t}}{(w + \rho S - S')^2} + \frac{\rho}{w + \rho S - S'}e^{-\alpha t} \]

Cancelling the repetitive terms,

\[ \rho = \frac{(\rho S' - S'\prime)}{w + \rho S - S'} + r \]
Upon analyzing this result, we can easily see that it is equivalent to
\[
\frac{dU'}{dt} = \rho - r,
\]
that is, the rate of growth of the marginal utility of consumption is the difference between the interest rate on savings and the utility discount rate. Thus, when the two interest rates become equal the growth rate of marginal utility is zero – the worker does not extract any additional utility from his flow of consumption. The left-hand term is also equal to \( \frac{dC}{dt} = \frac{C'}{C} \), that is,
\[
\frac{dC}{dt} = \rho - r
\]
or the rate of growth of consumption is the difference between the two interest rates. This means that the worker will have a positive consumption growth rate (his consumption will grow) when his savings grow faster than his utility. His consumption will decline if the utility discount rate exceeds the savings rate, that is, savings grow more slowly than utility. Finally, when the two rates are equal, the worker’s consumption will not grow at all. Solving the equation further,
\[
\frac{(\rho S' - S^*)}{w + \rho S - S^*} = \rho - r
\]
\[
\rho S' - S^* = (\rho - r)w + \rho(\rho - r)S - (\rho - r)S'
\]
\[
S'' - (2\rho - r)S' + \rho(\rho - r)S = (r - \rho)w
\]
We can solve this linear second-order differential equation in \( S(t) \) by the well-known steps. The particular integral is
\[
S_p = -\frac{w(\rho - r)}{\rho(\rho - r)} = -\frac{w}{\rho}
\]
\[
\begin{align*}
   r_{1,2} &= \frac{2\rho - r \pm \sqrt{(2\rho - r)^2 - 4(\rho^2 - \rho r)}}{2} \\
   &= \frac{2\rho - r \pm \sqrt{4\rho^2 - 4\rho r + r^2 - 4\rho^2 + 4\rho r}}{2} \\
   &= \frac{2\rho - r \pm \sqrt{r^2}}{2} \\
   &= \frac{2\rho - r \pm r}{2}
\end{align*}
\]
Hence, \( r_1 = \rho \) and \( r_2 = \rho - r \)
\[
S(t) = A_1e^{rt} + A_2e^{(\rho - r)t} - \frac{w}{\rho}
\]
Since at least one root \( (\rho) \) is positive, the time path of the savings function is dynamically unstable.

Using the endpoint conditions, we can find the arbitrary constants \( A_1 \) and \( A_2 \).

19. For the worker described in the previous example, assume that his utility function is
\[
U[C(t)] = [C(t)]^n. 
\]
Furthermore, the worker’s wage \( w \) is taxed with a unit tax rate \( \tau \). Maximize the discounted value of the worker’s utility function. What is the effect of the tax on the consumption and the savings function of the worker?
Solution:
This time, accounting for the tax rate, we have
\[ S'(t) = (1 - \tau)w + \rho S(t) - C(t) \]
Hence, the consumption function is
\[ C(t) = (1 - \tau)w + \rho S(t) - S'(t) \]
We optimize
\[
\max \int_0^T U[C(t)]e^{-\gamma t} dt = \int_0^T [(1 - \tau)w + \rho S(t) - S'(t)] e^{-\gamma t} dt
\]
where \( S(0) = S(T) = 0 \) and
\[ F[t, S(t), S'(t)] = [(1 - \tau)w + \rho S - S'] e^{-\gamma t} \]
\[ F_S = n \rho [(1 - \tau)w + \rho S - S']^{-1} e^{-\gamma t} \quad \text{and} \quad F_{S'} = -n[(1 - \tau)w + \rho S - S']^{-2} e^{-\gamma t} \]
Using the Euler’s equation,
\[
n \rho [(1 - \tau)w + \rho S - S']^{-1} e^{-\gamma t} = \frac{d}{dt}\left\{-n[(1 - \tau)w + \rho S - S']^{-1} e^{-\gamma t}\right\}
\]
\[ n \rho [(1 - \tau)w + \rho S - S']^{-1} e^{-\gamma t} = -n(n - 1)[(1 - \tau)w + \rho S - S']^{-2} (\rho S' - S')e^{-\gamma t} +
+ nr[(1 - \tau)w + \rho S - S']^{-1} e^{-\gamma t} \]
Simplifying,
\[ \rho = -(n - 1)[(1 - \tau)w + \rho S - S']^{-1} (\rho S' - S') + r \]
\[ \rho = \frac{(1 - n)(\rho S' - S')}{(1 - \tau)w + \rho S - S'} + r \]
\[ \frac{(1 - n)(\rho S' - S')}{(1 - \tau)w + \rho S - S'} = \rho - r \]
\[ (1 - n)\rho S' - (1 - n)S' = (\rho - r)(1 - \tau)w + \rho(\rho - r)S - (\rho - r)S' \]
\[ (1 - n)S' - (2\rho - \rho n - r)S' + \rho(\rho - r)S = (r - \rho)(1 - \tau)w \]
Normalizing the equation,
\[ S' = \frac{(2\rho - \rho n - r)}{1 - n} S' + \frac{(\rho^2 - \rho r)}{1 - n} S = \frac{(r - \rho)(1 - \tau)w}{1 - n} \]
\[ S' = \frac{(\rho - r)(1 - \tau)w(1 - n)}{(1 - n)\rho(\rho - r)} = -\frac{(1 - \tau)w}{\rho} \]
\[ \frac{(2\rho - \rho n - r)}{1 - n} \pm \sqrt{\frac{2\rho - \rho n - r}{1 - n}}^2 - 4\frac{(\rho^2 - \rho r)}{1 - n} \]
\[ r_{1,2} = \frac{2(\rho - \rho n - r) \pm \sqrt{(2\rho - \rho n - r)^2 - 4(\rho^2 - \rho r)(1 - n)}}{2(1 - n)} = \]
\[
\frac{2\rho - \rho n - r \pm \sqrt{(\rho n - r)^2}}{2(1-n)} = \frac{2\rho - \rho n - r \pm (\rho n - r)}{2(1-n)}
\]

Hence, \( r_1 = \rho \) and \( r_2 = \frac{\rho - r}{1-n} \) and the time path of the savings function is

\[
S(t) = A_1 e^{\rho t} + A_2 e^{(\rho - r) t} - \frac{(1-\tau)w}{\rho}
\]

Since at least one root (\( \rho \)) is positive, the time path of the savings function is divergent. The effect of the unit tax is to reduce the consumption of the worker, as can be seen from the consumption function, and change the equilibrium value of the savings function.

20. For the worker in problem 18, assume that his utility function is \( U[C(t)] = \frac{[(C(t))^{1-\eta}]}{1-\eta} \) so that the marginal utility of consumption is \( U' = C^{-\eta} \). Maximize the discounted value of the worker’s utility function. How does the result change when the worker has some initial assets to the amount of \( S_0 \) and wants to leave an inheritance of \( S_1 \), both positive?

Solution:
Again the consumption function is

\[
C(t) = w + \rho S(t) - S'(t)
\]

Thus,

\[
\max \int_0^T U[C(t)] e^{-\eta t} dt = \int_0^T \left[ \frac{[(w + \rho S(t) - S'(t))^{1-\eta}]}{1-\eta} \right] e^{-\eta t} dt
\]

where initially \( S(0) = S(T) = 0 \)

\[
F_t[S(t), S'(t)] = \frac{(w + \rho S - S')^{1-\eta}}{1-\eta} e^{-\eta t}
\]

\[
F_s = \rho(w + \rho S - S')^{-\eta} e^{-\eta t} \quad \text{and} \quad F_s' = -(w + \rho S - S')^{-\eta} e^{-\eta t}
\]

Using the Euler’s equation,

\[
\rho(w + \rho S - S')^{-\eta} e^{-\eta t} = \frac{d}{dt} \left[ -(w + \rho S - S')^{-\eta} e^{-\eta t} \right]
\]

\[
\rho(w + \rho S - S')^{-\eta} e^{-\eta t} = \eta(w + \rho S - S')^{-\eta-1} (\rho S' - S') e^{-\eta t} + r(w + \rho S - S')^{-\eta} e^{-\eta t}
\]

\[
\rho = \eta(w + \rho S - S')^{-1}(\rho S' - S') + r
\]

\[
\rho = \frac{\eta(\rho S' - S')}{w + \rho S - S'} + r \quad \text{which gives}
\]

\[
S' = \frac{(\rho + \rho \eta - r)}{\eta} S' + \frac{(\rho^2 - \rho r)}{\eta} S = \frac{(r - \rho)w}{\eta}
\]

\[
S_p = -\frac{(\rho - r)w \eta}{\eta \rho (\rho - r)} = -\frac{w}{\rho}
\]
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21. An individual wants to maximize the discounted value of his lifetime utility from consumption $U_1(C)$ as well as the utility from leaving a one time bequest to his children $U_2(S)$ at death. He obtains income $I$. The interest rate on savings is $\rho$, while the rate of discount is $r$. Express the individual’s flow budget constraint. Maximize his total utility.

Solution:

For the individual’s flow of income, we have

$$S'(t) = \rho S(t) + I - C(t)$$

where the increase in savings comes from the interest earned on it plus the share of income not spent on consumption. Therefore,

$$C(t) = \rho S(t) + I - S'(t)$$

The optimization problem is

$$\max_{S(t)} \int_0^T U_1(C(t))e^{-rt}dt + U_2(S(t))e^{-rt}$$

To maximize $\int_0^T U_1(C(t))e^{-rt}dt$, we use the calculus of variations, that is,
max \( \int_0^T U_i(C)e^{-rt} \, dt = \int_0^T U_i(\rho S + I - S) e^{-rt} \, dt \)

\[ F = U_1(\rho S + I - S) e^{-rt} \quad \text{and} \]

\[ F_S = U'_1 e^{-rt} \quad \text{and} \quad F_S = -U'_1 e^{-rt} \quad \text{where} \quad U'_1 = \frac{dU_1}{dC} \quad \text{and} \quad \text{from Euler's equation} \]

\[ F_S = \frac{dF_S}{dt} \quad \text{or} \]

\[ U'_1 e^{-rt} = \frac{d}{dt}(-U'_1 e^{-rt}) \]

\[ U'_1 e^{-rt} = -U''_1(\rho S' - S^*) e^{-rt} + rU'_1 e^{-rt} \quad \text{where} \quad U''_1 = \frac{d^2U_1}{dC^2} \]

\[ U'_1(\rho S' - S^*) \quad \text{or} \quad \frac{U''_1(\rho S' - S^*)}{U'_1} = r - \rho \]

\[ \frac{dU'_1}{dt} = r - \rho \]

We again obtain that the growth rate of the marginal utility of consumption is the difference between the interest rate on savings and the utility discount rate, so that when the two rates are equal, the individual does not extract any additional utility from further consumption.

We maximize the second term \( U_2(S)e^{-rt} \) separately, using the techniques of static optimization. By the first-order condition where we differentiate the term with respect to time, we get

\[ \frac{d}{dt}[U_2(S)e^{rt}] = U'_2 S' e^{-rt} - rU_2 e^{-rt} = 0 \quad \text{where} \quad U'_2 = \frac{dU_2}{dS} \]

\[ U'_2 S' = rU_2 \quad \text{Therefore,} \]

\[ \frac{U'_2 S'}{U_2} = r \quad \text{or} \]

\[ \frac{dU'_2}{dt} = r \]

This time we obtain that, at the optimum, the utility the individual extracts from leaving a bequest to his children grows exactly at the rate of discount \( r \). He will experience an increase in his utility if the real discount rate is positive. His utility will decline with a negative real discount rate. If the real rate happens to be zero, there will be no change in the individual’s utility. When the discount and the interest rate are the same, the individual would not extract any further utility from consumption; but the higher the interest rate, the faster his savings will grow and, hence, his utility from leaving a larger sum to his successors will increase. Thus, the individual will choose to substitute consumption with savings.

22. In his neoclassical growth model, Ramsey assumes that the savings rate is not constant but follows a particular time path. If the saving rate is \( s = s(t) \) (where \( 0 < s < 1 \)), a simple national-income model would be \( C = (1-s)Y(s) \) where national income \( Y(s) \) is itself a function of the savings rate.

Consumption is national income less aggregate savings. If instead \( Y(s) \), we assume that national
income depends also on the rate of change of the saving rate such that \( Y(s, s') \) and the nation experiences a general utility function \( U[C(t)] \), maximize the discounted value of aggregate utility from moment \( t_0 \) to moment \( t_1 \). Assume a discount rate of \( r \).

**Solution:**

We have

\[
\max_{t_0} \int_{t_0}^{t_1} U[C(t)] e^{-rt} dt = \int_{t_0}^{t_1} U[(1-s)Y(s, s')] e^{-rt} dt
\]

The integrand is

\[
F(t, s, s') = U[(1-s)Y(s, s')] e^{-rt}
\]

\[
F_x = U'[−Y + (1-s)Y_s] e^{-rt}
\]

\[
F_s = U'[1-s] e^{-rt}
\]

Following the Euler’s equation, we have \( F_s = \frac{dF_x}{dt} \) or

\[
\frac{d}{dt} \left[ U'\left[−Y + (1-s)Y_s\right] e^{-rt}\right] = \frac{d}{dt} \left[ U'\left[(1-s)Y_s\right] e^{-rt}\right]
\]

\[
U'\left[−Y + (1-s)Y_s\right] e^{-rt} = \frac{d}{dt} \left[ U'\left[(1-s)Y_s\right] e^{-rt}\right]
\]

\[
U'\left[−Y + (1-s)Y_s\right] e^{-rt} = U''[(-s')Y + (1-s)(Y_s' + Y_s s^s)]\left[(1-s)Y_s\right] e^{-rt} +
\]

\[
+ U'(-s'Y_s + (1-s)Y_s's^s) e^{-rt} - rU'[(1-s)Y_s] e^{-rt}
\]

where we assume \( U'' = \frac{d^2U}{dC^2} \) and \( Y_s' = \frac{\partial Y}{\partial s} \)

\[
−Y + (1-s)Y_s = \frac{U''}{U'}\left[(-s')Y + (1-s)(Y_s' + Y_s s^s)\right]\left[(1-s)Y_s\right] - s'Y_s + (1-s)Y_s's^s - r(1-s)Y_s;
\]

\[
\frac{U''}{U'}\left[(-s')Y + (1-s)(Y_s' + Y_s s^s)\right]\left[(1-s)Y_s\right] = -Y + (1-s)Y_s + s'Y_s - (1-s)Y_s's^s + r(1-s)Y_s
\]

\[
\frac{U''}{U'}\left[(-s')Y + (1-s)(Y_s' + Y_s s^s)\right] = \frac{-Y + s'Y_s + Y_s's^s}{(1-s)Y_s} + r
\]

If we assume \( s^s = 0 \), that is, the savings rate to be increasing at a constant rate \( s' \), we obtain

\[
\frac{U''}{U'}\left[(-s')Y + (1-s)Y_s s^s\right] = \frac{-Y + s'Y_s + (1-s)Y_s}{(1-s)Y_s} + r
\]

\[
\frac{dU'/dt}{U'} = \frac{-Y + s'Y_s + (1-s)Y_s}{(1-s)Y_s} + r
\]

The left side is nothing but the growth rate of the marginal utility of consumption. The assumption \( s' = const. \) implies that the economy is in a steady state and, therefore, in such a state the growth of marginal utility would depend positively on the discount rate. The higher the discount rate, the more stimulated people would feel to consume. An additional term showing the effect of the savings rate on national income influences the growth rate of marginal utility. Depending on the value of this term the growth rate could be equal to, higher or smaller than the discount rate. From the equation we can immediately see that the faster the savings ratio grows, that is, the higher \( s' \), the faster the marginal utility of consumption grows.
23. The consumption function of a country depends both on the marginal propensity to consume of the nation \( c(t) \) and its rate of change with time \( c'(t) \) such that \( C(t) = 7.5c^2 - 9c + 6(c')^2 \). Suppose that the goal of the government is to maximize total consumption over some time horizon \( [0,T] \). Find and analyze the optimal time path of the marginal propensity to consume.

**Solution:**

The optimization problem for the government is

\[
\max \int_0^T C(t)dt = \int_0^T \left[ 7.5c^2 - 9c + 6(c')^2 \right]dt
\]

where \( F[t,c(t),c'(t)] = 7.5c^2 - 9c + 6(c')^2 \) and \( F_c = 15c - 9 \) and \( F_{c'} = 12c' \). Furthermore, from the Euler’s equation,

\[
15c - 9 = \frac{d}{dt}(12c')
\]

\[
15c - 9 = 12c''
\]

\[
12c'' - 15c = -9
\]

\[
c'' - 1.25c = -0.75 \quad \text{which gives}
\]

\[
c_p = \frac{-0.75}{-1.25} = 0.6
\]

For the complementary function, we have

\[
r_{1,2} = \pm \sqrt{0 - 4(-1.25)} = \pm \sqrt{5}
\]

\[
c(t) = A_1e^{\sqrt{5}/2t} + A_2e^{-\sqrt{5}/2t} + 0.6
\]

With specific values for the endpoint conditions, we can definitize the arbitrary constants \( A_1 \) and \( A_2 \). The particular integral gives the intertemporal equilibrium value for the marginal propensity to consume. Here the value is 0.6, which means that, in equilibrium, 60% of the national income should go to consumption. But since we have one positive characteristic root, that is, \( r_1 = \frac{\sqrt{5}}{2} \), we have a divergent time path for the marginal propensity to consume.

24. Aggregate consumption \( C(t) \) is positively related to national income \( Y(t) \) in equilibrium as is consistent with economic theory. However, it also increases in times of economic growth, that is, when national income grows. Therefore, the specific form of the consumption function is

\[
C(t) = \alpha + \beta(1 - \tau)Y + \delta Y' \quad \alpha, \delta > 0 \quad 0 < \beta, \tau < 1
\]

where \( \alpha \) is autonomous consumption, \( \beta \) is the marginal propensity to consume, \( \tau \) is the percentage income tax rate, and \( \delta \) is a positive parameter that relates aggregate consumption to the rate of change of national income. Using the calculus of variations, maximize the utility from aggregate consumption \( U[C(t)] = \ln C(t) \) over the interval \( 0 \leq t \leq T \). Assume that national income is autonomous and independent of aggregate consumption.

**Solution:**

We have to solve
max \[ \frac{1}{\tau} \int_0^\tau [C(t)] dt = \int_0^\tau \ln[\alpha + \beta(1-\tau)Y + \delta Y'] dt \]

subject to \[ Y(0) = Y_o \quad \text{and} \quad Y(T) = Y_T \]

Thus, \[ F[t, Y(t), Y'(t)] = \ln[\alpha + \beta(1-\tau)Y + \delta Y'] \]

Thus, \[ F_Y = \frac{\beta(1-\tau)}{\alpha + \beta(1-\tau)Y + \delta Y'} \quad \text{and} \quad F_{Y'} = \frac{\delta}{\alpha + \beta(1-\tau)Y + \delta Y'} \]

\[ \frac{\beta(1-\tau)}{\alpha + \beta(1-\tau)Y + \delta Y'} = \frac{d}{dt} \left[ \frac{\delta}{\alpha + \beta(1-\tau)Y + \delta Y'} \right] \]

\[ \frac{\beta(1-\tau)}{\alpha + \beta(1-\tau)Y + \delta Y'} = -\delta \left[ (1-\tau)Y' + \beta Y'' \right] \quad \left[ \alpha + \beta(1-\tau)Y + \delta Y' \right]^2 \]

\[ \beta(1-\tau) = -\frac{\delta \left[ (1-\tau)Y' + \beta Y'' \right]}{\alpha + \beta(1-\tau)Y + \delta Y'} \]

\[ \alpha \beta(1-\tau) + \beta^2(1-\tau)^2Y + \beta \delta(1-\tau)Y' = -\beta \delta(1-\tau)Y' - \delta^2 Y'' \]

\[ \delta^2 Y'' + 2\beta \delta(1-\tau)Y' + \beta^2(1-\tau)^2Y = -\alpha \beta(1-\tau) \]

\[ Y'' = \frac{2\beta(1-\tau)}{\delta} \quad Y' + \frac{\beta^2(1-\tau)^2}{\delta^2}Y = -\frac{\alpha \beta(1-\tau)}{\delta^2} \]

\[ Y' = -\frac{\alpha \beta(1-\tau)}{\beta^2(1-\tau)^2} \]

\[ Y(t) = Ae^{\frac{\beta(1-\tau)}{\delta}} - \frac{\alpha}{\beta(1-\tau)} \]

25. The government of a country collects tax from the population that depends on actual national income \( Y(t) \) and the business cycle. When the economy experiences a boom, so that actual national income increases above its potential level, the government collects more tax in order to alleviate a preheated economy. On the contrary, in times of recession when aggregate output declines, the government reduces the amount of tax collected aiming to stimulate the economy. Thus, the tax function takes the form

\[ T[Y(t)] = \gamma + \tau Y + \delta Y' \quad \gamma, \delta > 0 \quad 0 < \tau < 1 \]

where \( \gamma \) is tax collected from nonincome sources, \( \tau \) is income tax, and \( \delta \) is a parameter relating tax policy to the business cycle. It is also known that the government wishes to optimize its expenditures where \( G[T(Y)] \). Maximize the government spending function, given the total tax collected. Assume that national income is autonomous and independent of any of the other variables. What is the optimal time path of national income with a specific form of the government spending function \( G(T) = \alpha \sqrt{T(Y)} \) where \( \alpha \) is a positive parameter?
Solution:

Since \( T[Y(t)] = \gamma + \tau Y + \delta Y' \), we have for government spending

\[
G[T(Y)] = G[\gamma +\tau Y(t) + \delta Y'(t)] \quad \text{and the optimization problem becomes}
\]

\[
\max_0^T \int_0^T G[T(Y)] \, dt = \int_0^T G[\gamma +\tau Y(t) + \delta Y'(t)] \, dt
\]

subject to \( Y(0) = Y_o \) and \( Y(T) = Y_r \)

where \( F[t, Y(t), Y'(t)] = G[\gamma +\tau Y(t) + \delta Y'(t)] \). From \( G[T(Y)] \) we also have

\[
G' = \frac{dG}{dT} \quad \text{and} \quad G^* = \frac{d}{dT} \left( \frac{dG}{dT} \right) \quad T' = \frac{dT}{dt} = \tau Y' + \delta Y^*
\]

From the integrand,

\[
F_y = G'[T(Y)]Y'(Y) = G'\tau \quad \text{and} \quad F_{yy} = G'[T(Y)]T'(Y') = G'\delta
\]

\[
G'\tau = \frac{d}{dt}(G'\delta)
\]

\[
G'\tau = \delta G^* T' = \delta G^*(\tau Y' + \delta Y^*)
\]

which gives a second-order ordinary differential equation in \( Y(t) \) or

\[
\delta Y^* + \tau Y' = \frac{\tau G'}{\delta G^*}
\]

\[
Y^* + \frac{\tau}{\delta} Y' = \frac{\tau G'}{\delta^2 G^*}
\]

From the specific form of the government spending function \( G(T) = \alpha \sqrt{T(Y)} \), we have

\[
G(T) = \alpha \sqrt{T(Y)} \quad \text{from which} \quad G' = \frac{\alpha}{2\sqrt{T}} \quad \text{and} \quad G^* = -\frac{\alpha}{4T\sqrt{T}} \quad . \quad \text{Substituting in the differential equation for} \quad Y ,
\]

\[
Y^* + \frac{\tau}{\delta} Y' = -\frac{\tau \alpha 4T \sqrt{T} \alpha}{\delta^2 2\sqrt{T} \alpha}
\]

\[
Y^* + \frac{\tau}{\delta} Y' = -\frac{2 \tau T}{\delta^2}
\]

\[
Y^* + \frac{\tau}{\delta} Y' = -\frac{2 \tau (\gamma + \tau Y + \delta Y')}{\delta^2}
\]

\[
Y^* + \frac{\tau}{\delta} Y' = -\frac{2 \tau \gamma}{\delta^2} - \frac{2 \tau^2 Y}{\delta^2} - \frac{2 \tau Y'}{\delta^2}
\]

\[
Y^* + \frac{3 \tau}{\delta} Y' + \frac{2 \tau^2 Y}{\delta^2} = -\frac{2 \tau \gamma}{\delta^2}
\]

\[
Y_p = -\frac{2 \tau \gamma}{\delta^2 2 \tau^2} = -\frac{\gamma}{\tau}
\]
Thus, the optimal time path of national income is

\[ Y(t) = A_1 e^{\frac{2\tau}{\delta}} + A_2 e^{\frac{-\tau}{\delta}} - \frac{\theta}{\tau} \]

Since both characteristic roots in the solution are negative, given the specific government spending function, the time path of national income is convergent. To specify the arbitrary constants, we can use the boundary conditions \( Y(0) = Y_o \) and \( Y(T) = Y_f \).

26. The stock of fish in a private lake is \( x(t) \) at any time \( t \) where the initial stock is \( x(t_o) = x_o \). It is assumed that the only value of the lake is the value of the fish in it. The owner is trying to use the lake sustainably by harvesting only the increase in the fish stock \( x'(t) \) resulting from the biological growth of the fish as a replenishable resource. Thus, \( x(t) \) is the natural, sustainable level of the stock. The fish is sold in the market at a constant price \( p \), and the costs of harvesting are given by the general function \( C[x'(t)] \). Furthermore, the owner has calculated that his unit cost of maintaining the entire fish stock is \( c \). He wishes to maximize the discounted value of his wealth between moments \( t_o \) and \( t_1 \) where the interest rate of discounting is \( r \). Formulate and solve generally the optimization problem for the lake owner.

**Solution:**

The wealth of the owner can be expressed as the total revenue minus the total costs. Hence,

\[ W[x(t)] = px'(t) - C[x'(t)] - cx(t) \]

where \( px'(t) \) is the total revenue from the lake. Total cost comprises the costs of harvesting \( C[x'(t)] \) and the costs of maintaining the resources, \( cx(t) \). Thus, the optimization problem for the owner is

\[ \max \int_{t_o}^{t_1} W[x(t)] dt = \int_{t_o}^{t_1} [px'(t) - C[x'(t)] - cx(t)] e^{-rt} dt \]

subject to \( x(t_o) = x_o \) and \( x(t_1) = x_1 \)

\[ F[t, x(t), x'(t)] = [px'(t) - C[x'(t)] - cx(t)] e^{-rt} \]

\[ F_x = -ce^{-rt} \quad F_{x'} = (p - C')e^{-rt} \quad \text{where} \quad C' = \frac{dC}{dx'} \]

and from the Euler’s equation,

\[ -ce^{-rt} = \frac{d}{dt} (p - C')e^{-rt} \]

\[ -ce^{-rt} = -C'x'e^{-rt} - r(p - C')e^{-rt} \]

\[ c = C'x'e^{-rt} + r(p - C') \]

which can be solved further with a specific form of the cost of harvesting function.
27. Consider a slightly modified model of fish harvesting where fish is harvested at the rate $u(t)$ and sold in the market at some constant price $p$. The fish stock in the lake is $x(t)$. It is also known that the rate of change of the stock of fish depends on some biological rate of growth $g(x)$ and the rate of harvest $u(t)$ such that $x'(t) = g[x(t)] - u(t)$. Furthermore, the costs of harvesting are given by the general function $C[u(t)]$, and the unit cost of maintaining the entire fish stock is $c$. Maximize the discounted value of the fish as a source of wealth between moments $t_o$ and $t_1$ where the rate of discount is $r$. What does the growth of fish depend on?

Solution:
This time, we have a wealth function

$$W = p u(t) - C[u(t)] - c x(t)$$

where from the growth equation we obtain $u(t) = g[x(t)] - x'(t)$. Hence, the wealth function can be expressed alternatively as

$$W = p[g(x) - x'] - C[g(x) - x'] - c x(t)$$

Now the optimization problem is

$$\max_{t \in [t_o, t_1]} \int_{t_o}^{t_1} W[x(t)] dt = \int_{t_o}^{t_1} \{ p[g(x) - x'] - C[g(x) - x'] - c x(t) \} e^{-rt} dt$$

subject to $x(t_o) = x_o$ and $x(t_1) = x_1$

$$F[t, x(t), x'(t)] = \{ p[g(x) - x'] - C[g(x) - x'] - c x(t) \} e^{-rt}$$

$$F_x = (pg' - C'g' - c)e^{-rt} \quad \quad F_{x'} = (p + C')e^{-rt}$$

where $g' = \frac{dg}{dx} \quad \quad C' = \frac{dC}{du} \quad \quad C'' = \frac{d}{du} \left( \frac{dC}{du} \right)$ and from the Euler’s equation,

$$(pg' - C'g' - c)e^{-rt} = \frac{d}{dt} \left[ (C' - p)e^{-rt} \right]$$

$$(pg' - C'g' - c)e^{-rt} = C''(g' - x^*)e^{-rt} + r(p - C')e^{-rt}$$

$$pg' - C'g' - c = C''(g' - x^*) + r(p - C')$$

which is a second-order differential equation in the state variable $x(t)$. It can be solved with a specific form of the cost and growth functions. However, some interesting results can be analyzed further by rearranging the equation

$$(p - C')g' = C''(g' - x^*) + c + r(p - C')$$

$$g'(t) = r - \frac{C''(x^* - g')}{p - C'} + \frac{c}{p - C'}$$

where $g'(t)$ is the rate of growth of the fish stock and represents the benefit of waiting and giving up the present consumption of fish. In static optimization, this marginal benefit of waiting would equal the opportunity cost of using the resource (that is, capital) in the present which is represented here by the interest rate $r$. However, in dynamic models, decisions in the present affect the future and the consumption of fish affects the rate of change of the marginal value of the fish. Thus, there is an additional loss, called capital loss, that must be added to the direct cost of waiting. If we let $p - C' = v$, then $v$ is the marginal current value of the stock and represents the net current benefit of
fish extraction. It is equivalent to price minus marginal cost where the opportunity costs of future events are capitalized into present decisions. It is easy to check that the derivative of \( p - C' \) with time is \( C'(x' - g') \). Hence,

\[
\frac{C'(x' - g')}{p - C'} = \frac{v'}{v}
\]

where \( \frac{v'}{v} \) is exactly the capital loss that must be added to the direct cost of waiting. Thus, the growth equation becomes

\[
g'(t) = r - \frac{v'}{v} + \frac{c}{v}
\]

Finally, the term \( \frac{c}{v} \) shows the effect on the cost of fishing of an increase in the fish stock since a larger stock of fish lowers the costs of fishing at any level of harvesting or extraction efforts. To summarize, the last equation gives a dynamic optimum condition – similar to nondynamic models, it equates the marginal benefit of an activity (on the left) to the marginal cost (on the right).

28. A firm utilizes some stock of capital \( K(t) \) at time \( t \) that generates a stream of rents for the firm \( R(K) \). Capital is assumed to depreciate at a linear rate \( \rho K(t) \) where \( \rho \) is a constant rate of depreciation of capital and \( 0 \leq \rho \leq 1 \). The firm appropriates new capital \( u(t) \), while the cost of investing in new capital is \( C(u) \). The firm wishes to maximize the discounted value of its wealth from capital accumulation over an infinite time horizon. Formulate the optimization problem and interpret the results. Assume a fixed market interest rate \( r \).

**Solution:**

For the change in the capital stock (or the investment rate), we have

\[
K'(t) = u(t) - \rho K(t),
\]

that is, the net increase in capital is the difference between new capital and the loss of old capital. Hence,

\[
u(t) = K'(t) + \rho K(t)
\]

The optimization problem for the firm is

\[
\max_{K} \int_{0}^{\infty} R(K) - C(K') e^{-rt} dt = \max_{K} \int_{0}^{\infty} [R(K) - C(K' + \rho K)] e^{-rt} dt
\]

\[
F[t, K(t), K'(t)] = [R(K) - C(K' + \rho K)] e^{-rt}
\]

\[
F_K = (R' - \rho C') e^{-rt}, \quad F_{K'} = -Ce^{-rt}
\]

where \( R' = \frac{dR}{dK} \), \( C' = \frac{dC}{du} \) and \( C* = \frac{d}{du} \left( \frac{dC}{du} \right) \)

and again following the Euler’s equation,

\[
(R' - \rho C') e^{-rt} = \frac{d}{dt} (-C'e^{-rt})
\]

\[
(R' - \rho C') e^{-rt} = rC'e^{-rt} - C'(K'' + \rho K') e^{-rt}
\]

\[
R' - \rho C' = rC' - C'(K'' + \rho K')
\]
If we rearrange slightly,

\[ R' + C''(K'' + \rho K') = (\rho + r)C' \quad \text{and} \]

\[ \frac{R'}{C'} + \frac{C''(K'' + \rho K')}{C'} = \rho + r \]

The derivative \( C'' \) here is the capital gain, that is, the rate of change in the marginal value of the capital occurring at time \( t \), which results from the future wealth-maximizing use of capital. This is an additional gain that attributes due to the optimal usage of the capital stock. We can just as well let \( C' = v \); the derivative of which with respect to time is the expression \( C''(K'' + \rho K') = v' \). Thus, the equation can be rewritten as

\[ \frac{R'}{v} + \frac{v'}{v} = \rho + r \]

We can interpret it in the following terms: the left-hand side illustrates the marginal benefits that result from the marginal profits of an additional increment of capital \( \frac{R'}{v} \), and the capital gain \( \frac{v'}{v} \) from the increase in the marginal value of capital used optimally. Clearly, on the right side we have the opportunity cost of capital, which involves the rate of depreciation of the capital stock plus the opportunities forgone on alternative investment \( r \). The equation can alternatively be written as

\[ R' + v' = (\rho + r)v \]

which denotes the same relationship, only in cumulative terms; for example, the term on the right side shows the cumulative opportunity cost of funds of the value of capital. In summary, the equation denotes the relationship between marginal revenue and cost, even though we are dealing with dynamics rather than statics. As with all dynamic models, though, the effect of current actions on future outcomes is taken into account.

29. For a firm with a total cost function \( TC = aq + b(q')^2 \), \((a, b > 0)\), where \( q(t) \) is the level of output and \( q'(t) \) is its rate of change, minimize the costs of production subject to the constraint

\[ \int_{t_o}^{t_1} q'(t)dt = N \]

where \( N \) is the number of units the firm should deliver at time \( t_1 \). Thus, the endpoint conditions are \( q(t_o) = 0 \) and \( q(t_1) = N \).

**Solution:**

The optimization problem is

\[ \min \int_{t_o}^{t_1} \left[ aq + b(q')^2 \right] dt \]

subject to

\[ \int_{t_o}^{t_1} q'(t)dt = N \]

where \( q(t_o) = 0 \) and \( q(t_1) = N \).

Following the Euler-Lagrangian method, we set the Lagrangian function

\[ \int_{t_o}^{t_1} \left[ aq + b(q')^2 + \lambda q' \right] dt \]
which gives the integrand \( L = aq + b(q')^2 + \lambda q' \). Hence, \( L_q = a \) and \( L_{q'} = 2bq' + \lambda \). From the Euler-Lagrange equation,

\[
a = \frac{d}{dt}(2bq' + \lambda)
\]

\[
a = 2bq^*
\]

\[
q^* = \frac{a}{2b}
\]

And the solution can be obtained through double integration. First,

\[
q' = \frac{at}{2b} + c_1
\]

and integrating once more,

\[
q(t) = \frac{at^2}{4b} + c_1t + c_2
\]

If we have precise values for the endpoint conditions, we can definitize the constants \( c_1 \) and \( c_2 \) and specify the time path of the output function.

30. For the firm in the preceding problem, minimize the discounted value of its total costs subject to the given constraint. Assume a discount rate of \( r \).

**Solution:**

The optimization problem is now

\[
\min \int_{t_0}^{t_1} \left[ aq + b(q')^2 \right] e^{-rt} dt
\]

subject to \( \int_{t_0}^{t_1} q'(t) dt = N \)

where \( q(t_0) = 0 \) and \( q(t_1) = N \)

Setting the Lagrangian,

\[
\int_{t_0}^{t_1} \left[ aq + b(q')^2 \right] e^{-rt} + \lambda q' dt
\]

So, the integrand is \( L = \left[ aq + b(q')^2 \right] e^{-rt} + \lambda q' \) where \( L_q = ae^{-rt} \) and \( L_{q'} = 2bq'e^{-rt} + \lambda \). Using the Euler-Lagrange equation,

\[
ae^{-rt} = \frac{d}{dt}(2bq'e^{-rt} + \lambda)
\]

\[
ae^{-rt} = 2bq^*e^{-rt} - 2rbq'e^{-rt}
\]

\[
a = 2bq^* - 2rbq'
\]

Rearranging and normalizing,

\[
q^* - rq' = \frac{a}{2b}
\]
Since \( a_1 = -r \), \( a_2 = 0 \) and \( c = \frac{a}{2b} \), the particular integral is
\[
q_p(t) = \frac{c}{a_1} t = -\frac{a}{2br} t
\]
\[
r_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} = \frac{r \pm \sqrt{r^2 - 4(0)}}{2} = r \pm r = 0, 1
\]

Therefore,
\[
q(t) = A_1 e^{rt} + A_2 - \frac{a}{2br} t
\]

To definitize the constants, we use the constraints \( q(t_0) \) and \( q(t_1) \).

31. A worker will receive a fixed wage rate \( w \) over his lifetime \( T \). He will also receive a constant interest rate \( \rho \) on his accumulated life savings \( S(t) \). The worker’s consumption flow is \( C(t) \), while his instantaneous utility function is \( U[C(t)] = \ln C(t) \). Furthermore, the worker receives an endowment of \( S_o \) from his ancestors and wishes to leave a bequest of \( S_1 \) to his children. If a fixed rate of discount \( r \) is applied, maximize the discounted value of the worker’s utility function subject to the constraint
\[
\int_0^T S'(t)dt = S_1 - S_o
\]

Solution:
The worker’s inflow of income is his wage plus the increase in his savings, that is, \( w + \rho S(t) \), while the outflow is his consumption. Therefore, the worker’s net flow of income is
\[
S'(t) = w + \rho S(t) - C(t)
\]

Hence, the consumption function is
\[
C(t) = w + \rho S(t) - S'(t)
\]

We need to optimize
\[
\max \int_0^T U[C(t)] e^{-\alpha t} dt = \int_0^T \ln[w + \rho S(t) - S'(t)] e^{-\alpha t} dt
\]
subject to \( \int_0^T S'(t)dt = S_1 - S_o \) where \( S(0) = S_o \) and \( S(T) = S_1 \)

The Lagrangian is
\[
\int_0^T \left\{ \ln[w + \rho S(t) - S'(t)] e^{-\alpha t} + \lambda S'(t) \right\} dt
\]
and the integrand is \( L = \ln[w + \rho S(t) - S'(t)] e^{-\alpha t} + \lambda S'(t) \). Hence,
\[
L_s = \frac{\rho}{w + \rho S - S'} e^{-\alpha t} \quad \text{and} \quad L_{s'} = -\frac{1}{w + \rho S - S'} e^{-\alpha t} + \lambda
\]

From the Euler-Lagrange equation,
Problems Book to Accompany Mathematics for Economists

\[ \frac{\rho}{w + \rho S - S'} e^{-\rho t} = \frac{d}{dt} \left[ -\frac{1}{w + \rho S - S'} e^{-\rho t} + \lambda \right] \]

\[ \frac{\rho}{w + \rho S - S'} e^{-\rho t} = \frac{(\rho S' - S^*)}{(w + \rho S - S')^2} e^{-\rho t} + \frac{r}{w + \rho S - S'} e^{-\rho t} \]

\[ \rho = \frac{(\rho S' - S^*)}{w + \rho S - S'} + r \quad \text{which gives} \]

\[ S^* - (2 \rho - r)S' + \rho (\rho - r)S = (r - \rho)w \]

\[ S_p = -\frac{w (\rho - r)}{\rho (\rho - r)} = -\frac{w}{\rho} \]

\[ r_{1,2} = \frac{2 \rho - r \pm \sqrt{(2 \rho - r)^2 - 4(\rho^2 - \rho r)}}{2} = \frac{2 \rho - r \pm \sqrt{4 \rho^2 - 4 \rho r + r^2 - 4 \rho^2 + 4 \rho r}}{2} = \frac{2 \rho - r \pm r}{2} \]

Hence, \( r_1 = \rho \) and \( r_2 = \rho - r \)

\[ S(t) = A_1 e^{\rho t} + A_2 e^{(\rho - r)t} - \frac{w}{\rho} \]

Since at least one root (\( \rho \)) is positive, the time path of the savings function is dynamically unstable.

From \( S(0) = S_o \) and \( S(T) = S_1 \), we also have

\[ S(0) = A_1 + A_2 - \frac{w}{\rho} = S_o \quad \text{so} \quad A_2 = S_o - A_1 + \frac{w}{p} \quad \text{and} \]

\[ S(T) = A_1 e^{\rho T} + A_2 e^{(\rho - r)T} - \frac{w}{\rho} = S_1 \]

\[ A_1 e^{\rho T} + \left( S_o - A_1 + \frac{w}{p} \right) e^{(\rho - r)T} - \frac{w}{\rho} = S_1 \]

\[ A_1 e^{\rho T} \left( 1 - e^{-rT} \right) + \left( S_o + \frac{w}{p} \right) e^{(\rho - r)T} = S_1 + \frac{w}{p} \]

\[ A_1 = \frac{S_1 + \frac{w}{p} - \left( S_o + \frac{w}{p} \right) e^{(\rho - r)T}}{e^{\rho T} \left( 1 - e^{-rT} \right)} \quad \text{and} \quad A_2 = \frac{\left( S_o + \frac{w}{p} \right) e^{\rho T} - S_1 - \frac{w}{p}}{e^{\rho T} \left( 1 - e^{-rT} \right)} \]

32. For the worker in the previous problem, assume his utility function is \( U[C(t)] = \frac{[C(t)]^{1-\eta}}{1-\eta} \).

Maximize the discounted value of the worker’s utility function. Discuss the growth rate of the worker’s consumption. Express the time path of his asset function, given the endpoint conditions.

**Solution:**

Again the consumption function is

\[ C(t) = w + \rho S(t) - S'(t) \]
Chapter 12. Introduction to Dynamic Optimization: The Calculus of Variations

\[
\max \int_0^T U[C(t)] e^{-rt} dt = \int_0^T \left[ \frac{(w + \rho S(t) - S'(t))^{1-\eta}}{1-\eta} \right] e^{-rt} dt
\]
subject to \( \int_0^T S'(t) dt = S_1 - S_o \)

where \( S(0) = S_o \) and \( S(T) = S_i \)

Given the asset constraint, the Lagrangian is

\[
L = \frac{(w + \rho S - S')^{1-\eta}}{1-\eta} e^{-rt} + \lambda S'
\]

\[
L_S = \rho(w + \rho S - S')^{-\eta} e^{-rt}
\]

\[
L_{S'} = -(w + \rho S - S')^{-\eta} e^{-rt} + \lambda
\]

\[
\rho = \eta(w + \rho S - S')^{-1}(\rho S' - S^*) + r
\]

\[
\frac{\rho - r}{\eta}
\]

Rearranging,

\[
\frac{(\rho S' - S^*)}{w + \rho S - S'} = \frac{\rho - r}{\eta}, \text{ that is,}
\]

\[
\frac{\rho}{C} = \frac{\rho - r}{\eta}
\]

An optimal time path dictates that the growth rate of the worker’s consumption be equal to the product

of the intertemporal elasticity of substitution \( \frac{1}{\eta} \) and the difference of the interest rate and the rate of

intertemporal discount. Here, \( \rho \) is the reward to the worker for his patience in postponing

consumption. At the same time, \( r \) is the cost of consuming in the present and works as a punishment

for the worker if he is eager to consume now. Such a great impatience reduces the rate of growth of

the worker’s consumption. Thus, the savings rate stimulates the growth rate of consumption, while the

rate of discount reduces it. Finally, a high elasticity of substitution also stimulates the growth of

consumption and implies that future consumption is a good substitute for current consumption.

\[
S^* - \frac{(\rho + \rho \eta - r)}{\eta} S' + \frac{(\rho^2 - \rho r)}{\eta} S = \frac{(r - \rho)w}{\eta}
\]

\[
S_p = -\frac{(\rho - r)\eta}{\eta \rho (\rho - r)} = \frac{-w}{\rho}
\]

\[
\frac{(\rho + \rho \eta - r)}{\eta} \pm \sqrt{\frac{\rho + \rho \eta - r}{\eta} - 4(\rho^2 - \rho r)} = \frac{\rho + \rho \eta - r \pm \sqrt{(\rho - \rho \eta - r)^2}}{2\eta} = \rho + \rho \eta - r \frac{\pm \sqrt{(\rho - \rho \eta - r)^2}}{2\eta}
\]
\[
\frac{\rho + \rho \eta - r \pm (\rho - \rho \eta - r)}{2\eta}
\]

\[r_1 = \rho \quad \text{and} \quad r_2 = \frac{\rho - r}{\eta}\]

and the time path of the savings function is

\[S(t) = A_1 e^{\rho t} + A_2 e^{\eta t} - \frac{w}{\rho}\]

From the endpoint conditions,

\[S(0) = A_1 + A_2 - \frac{w}{\rho} = S_o \quad \text{and} \quad A_2 = S_o - A_1 + \frac{w}{p}\]

\[S(T) = A_1 e^{\rho T} + A_2 e^{\eta T} - \frac{w}{\rho} = S_1\]

\[A_1 e^{\rho T} + \left( S_o - A_1 + \frac{w}{p} \right) e^{\eta T} - \frac{w}{\rho} = S_1\]

\[A_1 \left( e^{\rho T} - e^{\eta T} \right) + \left( S_o + \frac{w}{p} \right) e^{\eta T} = S_1 + \frac{w}{p}\]

\[A_1 = \frac{S_1 + \frac{w}{p} - \left( S_o + \frac{w}{p} \right) e^{\eta T}}{e^{\rho T} - e^{\eta T}} \quad \text{and} \quad A_2 = \frac{S_o e^{\rho T} - S_1 - \frac{w}{p} + \frac{w}{p} e^{\eta T}}{e^{\rho T} - e^{\eta T}}\]